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SOME CHARACTERIZATIONS OF FINITE GROUPS IN WHICH SEMIPERMUTABILITY IS A TRANSITIVE RELATION

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ABSTRACT. Let G be a finite group and let H be a subgroup of G . H is said to be *semipermutable* in G if H permutes with every subgroup K of G with $(|H|, |K|) = 1$. A number of new characterizations of finite solvable BT-groups are given, where a BT-group is one in which semipermutability is a transitive relation.

1. INTRODUCTION

All groups considered in this paper are finite. A subgroup H of a group G is said to *permute* with a subgroup K if HK is a subgroup of G . H is said to be *permutable* (*S-permutable*) in G if it permutes with all the subgroups (Sylow subgroups) of G . Subgroups H and K of G are *mutually permutable* provided that H permutes with every subgroup of K and K permutes with every subgroup of H . Further, H and K are said to be *totally permutable* in G if every subgroup of H permutes with every subgroup of K . Mutually permutable and totally permutable products have been studied in [5, 6, 7]. H is called *semipermutable* in G if H permutes with every subgroup K of G for which $(|H|, |K|) = 1$, and *S-semipermutable* in G if H permutes with every Sylow p -subgroup of G for which $(p, |H|) = 1$.

By a result of Kegel [10], an S -permutable subgroup of a group G is subnormal. This is not the case for S -semipermutability and semipermutability since a Sylow two subgroup of D_{10} , the dihedral group of order ten, is semipermutable and hence S -semipermutable but not subnormal. Semipermutable and S -semipermutable subgroups have been studied in [13, 14].

A group G is called a *PST-group* if S -permutability is a transitive relation in G , that is, if a subgroup H that is S -permutable in an S -permutable subgroup K of G is S -permutable in G . Solvable PST-groups have been studied in [1, 3, 4, 8]. By Kegel's result [10] (mentioned above), PST-groups are exactly those groups in which all subnormal subgroups are S -permutable.

Agrawal [1] introduced and classified solvable PST-groups. Let G be a solvable group with nilpotent residual L . Agrawal [1] showed that G is a solvable PST-group if and only if L is a normal abelian Hall subgroup on which G acts by conjugation as power automorphisms. The class of solvable PST-groups is both subgroup and quotient closed.

A group G is called a BT-group (see [13]) if semipermutability is a transitive relation in G . In a very interesting paper, Yangming Li, Lifang Wang, and Yanming Wang [13] prove the following theorem about solvable BT-groups.

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Theorem 1. [13] *Let G be a group with nilpotent residual L . The following statements are equivalent:*

- (1) G is a solvable BT-group;
- (2) every subgroup of G of prime power order is semipermutable in G ;
- (3) every subgroup of G is semipermutable in G ;
- (4) G is a solvable PST-group and if p and q are distinct primes not dividing the order of L with $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$, then $[G_p, G_q] = 1$.

Note that the class of solvable BT-groups is subgroup and quotient closed. Our purpose here is to establish several new characterizations of solvable BT-groups which includes a local characterization.

Theorem A. *Let G be a solvable group and D a system normalizer of G . G is a BT-group if and only if the following conditions are satisfied:*

- (1) If p and q are prime divisors of the order of D with $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$, then $[G_p, G_q] = 1$.
- (2) The subnormal subgroups of G are S -permutable in G .

A set of pairwise permutable Sylow subgroups of a group G , one for each prime dividing the order of G , is called a *Sylow basis* of G . Let $\{p_1, p_2, \dots, p_r\} = \pi(G)$, the set of all prime divisors of the order of G , and let $\{P_1, P_2, \dots, P_r\}$ be a Sylow basis of G with $P_i \in \text{Syl}_{p_i}(G)$ for all i . If θ is a subset of $\pi(G)$, then $H = \prod_{p_i \in \theta} P_i$ is a Hall θ -subgroup of G ; that is, $([G : H], |H|) = 1$. P. Hall (see 9.1.7 and 9.1.8 of [12]) showed that a group is solvable if and only if it has a Sylow basis. He also showed that in a solvable group G any two Sylow bases are conjugate and the Hall θ -subgroups form a unique conjugacy class of subgroups.

Let $\Gamma = \{P_1, P_2, \dots, P_r\}$ be a Sylow basis for a group G . Then $G = P_1 P_2 \cdots P_r$ is a product of pairwise permutable nilpotent groups. If the subgroups P_1, P_2, \dots, P_r are pairwise mutually permutable, then Γ is called a *strong Sylow basis*. If G has a strong Sylow basis, then it is supersolvable by Theorem 16 of [2]. Note that in a strong Sylow basis the elements are in fact pairwise totally permutable. This follows from the fact that the mutually permutable product of two groups with trivial intersection is a totally permutable product by part (iv) of Lemma 1 of [5].

A set of Sylow subgroups of a group G with exactly one Sylow subgroup for each prime dividing $|G|$ is called a *complete set of Sylow subgroups* of G . Let $\{p_1, p_2, \dots, p_k\}$ be the set of prime divisors of the order of a group G and let $\Sigma = \{P_1, P_2, \dots, P_k\}$ be a complete set of Sylow subgroups of G with $P_i \in \text{Syl}_{p_i}(G)$. Let N be a normal subgroup of G . Let us call Σ a *strong Sylow system of G with respect to N* if $P_i P_j$ is a mutually permutable product for all $P_i \in \Sigma$ with $p_i \in \pi(N)$. Note that it is not clear from the definition that a group G with a strong Sylow system Σ is solvable or has a strong Sylow basis. However, the following theorem is established in [8].

Theorem 2. [8] *Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N . The following statements are equivalent:*

- (1) G has a strong Sylow basis;
- (2) G has a strong Sylow system with respect to N ;
- (3) G is a solvable PST-group.

Again, let $\{p_1, p_2, \dots, p_k\}$ be the set of prime divisors of the order of a group G , let $\Sigma = \{P_1, P_2, \dots, P_k\}$ be an arbitrary complete set of Sylow subgroups of

G with $P_i \in \text{Syl}_{p_i}(G)$, and let N be a normal subgroup of G . We say G satisfies condition α with respect to N if Σ is a strong Sylow system with respect to N with the additional condition that $P_i P_j$ is a subgroup for all $p_i, p_j \notin \pi(N)$. If G satisfies condition α with respect to $N = G$, then we say that G satisfies condition α .

Theorem B. *Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N . The following statements are equivalent:*

- (1) G satisfies condition α ;
- (2) G satisfies condition α with respect to N ;
- (3) G is a solvable BT-group.

Remark. From Theorems 2 and B, we see that solvable PST-groups are those groups for which some complete set of Sylow subgroups is a Strong Sylow basis while the solvable BT-groups are those groups for which any arbitrary complete set of Sylow subgroups is a Strong Sylow basis.

Example 1. *Let $H = \langle x, y | x^3 = y^2 = 1, x^y = x^{-1} \rangle$ so that H is isomorphic to S_3 . Let $G = H \times Z_5$ where $Z_5 = \langle t \rangle$ is a cyclic group of order 5. Then $\langle xy \rangle, \langle y \rangle$ and $\langle yx \rangle$ are the Sylow 2-subgroups of G and $\{\langle t \rangle, \langle x \rangle, \langle y \rangle\}, \{\langle t \rangle, \langle x \rangle, \langle yx \rangle\}$ and $\{\langle t \rangle, \langle x \rangle, \langle xy \rangle\}$ are all the Sylow bases of G and they are all strong. Note that these Sylow bases are the only complete sets of Sylow subgroups of G . Thus G satisfies α . Also G is a PST-group which is also a BT-group.*

Example 2. *Let $G = \langle x, y, z | x^{11} = y^5 = z^2 = [y, z] = 1, x^y = x^3, x^z = x^{-1} \rangle$. The nilpotent residual G is $\langle x \rangle$ and G is a solvable PST-group by a result of Agrawal [1]. Note that $\langle y^x \rangle \in \text{Syl}_5(G)$, $\langle z \rangle \in \text{Syl}_2(G)$. But $\langle y^x \rangle \langle z \rangle \neq \langle z \rangle \langle y^x \rangle$ so that $\{\langle x \rangle, \langle y^x \rangle, \langle z \rangle\}$ is not a Sylow basis. Thus G does not satisfy property α and is not a BT-group.*

Another characterization of a solvable BT-group is the following:

Theorem C. *A group G is a solvable BT-group if and only if it has a normal subgroup N such that N and G/N'' are solvable BT-groups.*

Let p be a prime. A group G is called a C_p -group if every subgroup of a Sylow p -subgroup P of G is normal in $N_G(P)$. Robinson [11] showed that a finite group G is a solvable group with a transitive normality relation if and only if it is a C_p -group for all primes p . A group G is called a Y_p -group if whenever K is a p -subgroup of G every subgroup of K is S -permutable in $N_G(K)$. Ballester-Bolinches and Esteban-Romero [4] proved the following.

Theorem 3. [4] *Let G be a group.*

- (1) G is a solvable PST-group if and only if G is a Y_p -group for all primes p .
- (2) G is a Y_p -group if and only if G is either p -nilpotent or G is a C_p -group and has abelian Sylow p -subgroups.

Part (1) of Theorem 4 is a local characterization of solvable PST-groups. One of our purposes here is to present a local characterization for solvable BT-groups.

Let p be a prime. We introduce the following classes of groups:

Definition 1.

- (1) A group G is a \widehat{Y}_p -group if for every p -subgroup K of G every subgroup of K is semipermutable in $N_G(K)$.

- (2) A group G is a \widetilde{Y}_p -group if for every p -subgroup K of G every subgroup of K is S -semipermutable in $N_G(K)$.
- (3) A group G is a \overline{Y}_p -group if for every p -subgroup K of G , $N_G(K)$ has a strong Sylow basis.
- (4) A group G is a $\overline{\overline{Y}}_p$ -group if for every p -subgroup K of G , $N_G(K)$ satisfies property α .

We arrive at the following results.

Theorem D. Let p be a prime then $\overline{\overline{Y}}_p \subsetneq \overline{Y}_p \subsetneq \widehat{Y}_p = \widetilde{Y}_p = Y_p$.

Using Theorem 4, the following corollary is immediate.

Corollary A. Let G be a group. Then the following are equivalent:

- (1) G is a solvable PST-group.
- (2) G is a \widetilde{Y}_p -group for all primes p .
- (3) G is a \widehat{Y}_p -group for all primes p .

Theorem E. Let G be a group.

- (1) G is a solvable BT-group if and only if G is a $\overline{\overline{Y}}_p$ -group for all primes p .
- (2) G is a solvable PST-group if and only if G is a \overline{Y}_p -group for all primes p .

2. PROOFS OF THE THEOREMS

We first provide several lemmas which are used in the proofs of Theorems A-E.

Lemma A. Let G be a group with normal Sylow subgroups $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ with p and q distinct primes. If G/P and G/Q are solvable PST-groups, then G is also a solvable PST-group.

Proof. Note first that G is solvable, in fact supersolvable, and the nilpotent residual L of G is nilpotent. By Theorem 2.3 of [1] LP/P (resp. LQ/Q) is an abelian normal Hall subgroup of G/P (resp. G/Q) and G acts as power automorphisms on LP/P (resp. LQ/Q). Note that $L/P \cap L$ and $L/Q \cap L$ are abelian so that L is abelian. Assume that $(p, |L|) = 1$. Then LP/P is G -isomorphic to L and L is an abelian Hall subgroup of G . Also G acts as power automorphisms on L and G is a solvable PST-group by Theorem 2.5 of [1]. Thus we may assume both p and q divide $|L|$. It is clear that LP and LQ are Hall subgroups of G so that $Q \leq LP$ and $P \leq LQ$. Hence $PQ \leq L$. Now PQ/P is G -isomorphic to Q and so G acts as power automorphisms on Q . But G/Q is a solvable PST-group and hence by Theorem 2.5 of [1] G is a solvable PST-group. \square

Corollary B. Let G be a group with normal Sylow subgroups $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ with p and q distinct primes. If G/P and G/Q are solvable T-groups (PT-groups), then G is a solvable T-group (PT-group).

Proof. Assume G/P and G/Q are solvable T-groups. Then the Sylow subgroups of G/P and G/Q are Dedekind (Iwasawa) so that all the Sylow subgroups of G are Dedekind (Iwasawa). By Lemma A, G is a solvable PST-group so that G is a T-group (PT-group) by Theorem 3.2 (Theorem 3.1) of [1]. \square

Lemma B. Let G be a group with normal Sylow subgroups $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ with p and q distinct primes. If G/P and G/Q are solvable BT-groups, then G is a solvable BT-group.

Proof. By Theorem 1 and Lemma A, G is a solvable PST-group. Let L be the nilpotent residual of G . Let r and s be primes not in $\pi(L)$ and let $R \in \text{Syl}_r(G)$ and $S \in \text{Syl}_s(G)$. By part (4) of Theorem 1 it is enough to show $[R, S] = 1$. LP/P and LQ/Q are, respectively, the nilpotent residuals of G/P and G/Q , and note that $\pi(L)$ contains both $\pi(LP/P)$ and $\pi(LQ/Q)$. So, by Theorem 1 we have $[RP/P, SP/P] = 1$ and $[RQ/Q, SQ/Q] = 1$. Thus $[R, S] \leq Q \cap P = 1$ and the proof is complete. \square

Proof of Theorem A. Let G be a solvable BT-group. By part (4) of Theorem 1, G satisfies (1) and (2).

Conversely, assume G satisfies (1) and (2). Then, G is a solvable group and by (2) it is a PST-group. Let L be the nilpotent residual of G . By Theorem 2.3 of [1], L is a normal abelian Hall subgroup of G . Note that $G = LD$ and $L \cap D = \langle 1 \rangle$ by a result of Gaschütz, Schenkman and Carter (see Theorem 9.2.7 of [12]). Let p and q be distinct prime divisors of the order of D with $G_p \in \text{Syl}_p(G)$ and $G_q \in \text{Syl}_q(G)$. Then $[G_p, G_q] = 1$ by (1). Therefore, by Theorem 1, G is a BT-group. \square

Proof of Theorem B. If G satisfies condition α then it clearly satisfies condition α with respect to any normal subgroup N of G . So (1) implies (2).

We now show that (2) implies (3). We have that N is a normal subgroup of G such that G satisfies condition α with respect to N and G/N is nilpotent. We may assume $N \neq 1$. Note that G has a strong Sylow system with respect to N and so G is a solvable PST-group by Theorem A of [8].

Let L be the nilpotent residual of G . By Theorem 2.3 of [1], L is a normal abelian Hall subgroup of G . We may assume $L \neq 1$. Let $\pi = \pi(L)$ and let q and r be distinct primes contained in π' . Further let $R \in \text{Syl}_r(G)$ and $Q \in \text{Syl}_q(G)$. Since G satisfies condition α with respect to N it follows that QR is a subgroup of G . Now G/L is nilpotent and so QR is nilpotent. Thus $[Q, R] = 1$ and so G is a solvable BT-group by part (4) of Theorem 1. Therefore, (3) is a consequence of (2).

Finally, we show that (3) implies (1). Let G be a solvable BT-group. By part (4) of Theorem 1, G is a solvable PST-group. Let L be the nilpotent residual of G . By Theorem 2.3 of [1], L is a normal abelian Hall subgroup of G on which G acts by conjugation as power automorphisms. In particular, every subgroup of L is normal in G .

Let $\pi = \pi(L) = \{p_1, p_2, \dots, p_r\}$ and let $\pi' = \{p_{r+1}, \dots, p_t\}$. Let $P_i \in \text{Syl}_{p_i}(G)$ so that $\Sigma = \{P_1, P_2, \dots, P_r, P_{r+1}, \dots, P_t\}$ is an arbitrary complete set of Sylow subgroups of G . By part (4) of Theorem 1, $[P_j, P_k] = 1$ for $r+1 \leq j, k \leq t$ and the same is true when $1 \leq j, k \leq r$. Now consider $j \in \pi$ and $k \in \pi'$. Since every subgroup of P_j is normal in G , it follows that $P_j P_k$ is a pairwise mutually permutable pair. This shows Σ is a strong Sylow basis for G and G satisfies condition α . Therefore, (1) is a consequence of (3). \square

Proof of Theorem C. Let N be a normal subgroup of the group G . Assume that N and G/N'' are solvable BT-groups. We are to show that G is a solvable BT-group. By Theorem 1 and Theorem B of [8], G is a solvable PST-group. Note that if M is a normal subgroup of G , then G/M satisfies the assumptions of the theorem.

Let L be the nilpotent residual of G and let $\pi = \pi(L)$. By Theorem 2.3 of [1], L is a normal abelian Hall subgroup of G . Using induction on the order of G , we will

show that $G' = L = \text{Fit}(G)$ and so $G'' = L' = 1$ which means that $N'' = 1$ and so G is a solvable BT-group.

Assume that G has two distinct normal Sylow subgroups, say P and Q . Then, using induction, G/P and G/Q are solvable BT-groups and, by Lemma B, G is also a solvable BT-group. Thus we may assume that $\pi(L) = \{p\}$ where p is the largest prime divisor of $|G|$. Since G is supersolvable G' is nilpotent and $L \leq G' \leq \text{Fit}(G)$.

Let r and s be prime divisors of $|G|$ different from p and let $R \in \text{Syl}_r(G)$ and $S \in \text{Syl}_s(G)$. By Theorem 1, part (4), it is enough to show $[R, S] = 1$.

Assume M is a minimal normal subgroup of G whose order is a prime t , $t \neq p$. Then G/M is a solvable BT-group and LM/M is the nilpotent residual of G/M . By part (4) of Theorem 1, it follows that $[R, S] \leq M$. Similarly, $[R, S] \leq P$ and so $[R, S] \leq P \cap M = 1$. Hence, we may assume that $P = L \leq G' \leq \text{Fit}(G) = P$ and so $G' = L$. This means that $N'' \leq G'' = L' = 1$ so that G is a solvable BT-group. This completes the proof. \square

Proof of Theorem D. That $Y_p \subseteq \tilde{Y}_p$ is clear from the definitions. Let $G \in \tilde{Y}_p$ and let $H \leq K \leq P \in \text{Syl}_p(G)$. If $N_p \in \text{Syl}_p(N_G(K))$ then $H \leq K \trianglelefteq N_p$. So H clearly permutes with the Sylow p -subgroups of $N_G(K)$. Hence H is S-permutable in $N_G(K)$ and $G \in Y_p$.

Note it is clear from the definitions that $\hat{Y}_p \subseteq \tilde{Y}_p$.

Now let $G \in Y_p$. Let $H \leq K \leq P \in \text{Syl}_p(G)$ and let $N = N_G(K)$. By part (2) of Theorem 4, G is either p -nilpotent or $G \in C_p$ with abelian Sylow p -subgroups.

Suppose first that G is p -nilpotent. Then N is p -nilpotent. Let M be the normal Hall p' -subgroup of N . Then MK is a subgroup and $MK = M \times K$ is a direct product. Let L be a subgroup of N for which $(|L|, p) = 1$. Then L is a subgroup of M (this follows from 9.1.6 of [12] since p -nilpotence implies p' -separability). Hence $[L, H] \leq [M, K] = 1$. So $LH = HL$ and we have that H is semipermutable in N giving us that $G \in \hat{Y}_p$.

Now assume $G \in C_p$ with P abelian. Suppose that N is proper in G . Then since $N \in Y_p$, by induction, $N \in \hat{Y}_p$. Thus H is semipermutable in $N_N(K) = N_G(K)$. So we can assume $N = G$, that is, we can assume K is normal in G . Now, since $G \in Y_p$, we have H is S-permutable in $N = G$. Hence H is subnormal in G . If Q is a Sylow q -subgroup of G with $q \neq p$, then $HQ = QH$ and hence Q normalizes H . Now recall that P is abelian and so P normalizes H as well. So H is normal in G and so clearly H is semipermutable in N giving us $G \in \hat{Y}_p$. Hence $\hat{Y}_p = \tilde{Y}_p = Y_p$.

Now we will argue $\bar{Y}_p \subsetneq Y_p$. Assume that G is a \bar{Y}_p -group. Let $H \leq K$ where K is a p -subgroup of G and let $\{P_1, P_2, \dots, P_r\}$ be a strong Sylow basis for $N_G(K)$. We may assume that P_1 is a Sylow p -subgroup of $N_G(K)$ so that $H \leq K \leq P_1$. Since $P_1 P_i$ is a mutually permutable product, H permutes with each P_i . Now let Q be a Sylow subgroup of $N_G(K)$. Let $x \in N_G(K)$ and note that $\{P_1^x, P_2^x, \dots, P_r^x\}$ is a strong Sylow basis for $N_G(K)$. As above, $H \leq K \leq P_1^x$ and H permutes with P_j^x (for all j). Thus $HQ = QH$. This means that H is S-permutable in $N_G(K)$ and so G is a Y_p -group. Let $p = 7$ and put $W = C_7 \times A_5$ where A_5 is the simple group of order 60. Then W does not have a Sylow basis but satisfies Y_7 . Also $C_7 \times A_4$ is the alternating group of degree 4, satisfies Y_7 , and has a Sylow basis but not a strong Sylow basis. Hence $\bar{Y}_p \subsetneq Y_p$.

It is clear that $\bar{\bar{Y}}_p \subseteq \bar{Y}_p$ since a group satisfying property α certainly has a strong Sylow basis. That the inclusion is proper can be seen from a few examples. Let H

be any solvable PST-group with p not dividing the order of H with the additional assumption that H is not a BT-group. Let $G = H \times Z_p$ with Z_p cyclic of order p . Then G is a solvable PST-group and so are its subgroups. Hence for any p -subgroup K of G we have $N_G(K)$ is a PST-group and so by Theorem 2, we have $G \in \overline{Y}_p$. Now $Z_p \trianglelefteq G$ but G does not satisfy property α since G is not a BT-group. Hence G is not a \overline{Y}_p -group. For any prime p different from 11, 5, and 2, let H be the group in Example 2. For $p = 11$ or $p = 5$, let $H = \langle x, y, z \mid x^7 = y^3 = z^2 = [y, z] = 1, x^y = x^2, x^z = x^{-1} \rangle$. For $p = 2$, let $H = \langle x, y, z \mid x^{31} = y^5 = z^3 = [y, z] = 1, x^y = x^{16}, x^z = x^{25} \rangle$. This completes the proof. \square

Proof of Theorem E. To show (1), assume that \overline{G} is a solvable BT-group and let p be a prime. We are to show that G is a \overline{Y}_p -group. Let K be a p -subgroup of G . Since the class of solvable BT-groups is subgroup closed, we may assume $N_G(K) = G$. By Theorem B, G satisfies condition α and so G is a \overline{Y}_p -group.

Conversely, assume that G is a \overline{Y}_p -group for all primes p . By Theorem D, $\overline{Y}_p \subseteq \overline{Y}_p \subseteq Y_p$ for all p . Hence, by Theorem 4, G is a solvable PST-group. Let L be the nilpotent residual of G and let $\pi = \pi(L)$. We may assume $L \neq 1$ since a nilpotent group is a BT-group. Let $r \in \pi(L)$ and let p and q be distinct primes from π' . Let $R \in \text{Syl}_r(G)$, $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$. L is a normal abelian Hall subgroup of G and so $N_G(R) = G$. Since G is a \overline{Y}_r -group $\{P, Q, R\}$ is part of a strong Sylow basis of G and so PQ is a subgroup of G . But G/L is nilpotent so that $[P, Q] = 1$. By Theorem 1 G is a solvable BT-group.

Part (2) follows from Theorem 4 and from the fact that $\overline{Y}_p \subseteq Y_p$. \square

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