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Solvable PST-groups, strong Sylow bases and mutually permutable products

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ABSTRACT

A subgroup H of a group G is said to *permute* with the subgroup K of G if $HK = KH$. Subgroups H and K are *mutually permutable* (*totally permutable*) in G if every subgroup of H permutes with K and every subgroup of K permutes with H (if every subgroup of H permutes with every subgroup of K). If H and K are mutually permutable and $H \cap K = 1$, then H and K are totally permutable. A subgroup H of G is *S-permutable* in G if H permutes with every Sylow subgroup of G. A group G is called a *PST-group* if S-permutability is a transitive relation in G. Let $\{p_1, \dots, p_n, p_{n+1}, \dots, p_k\}$ be the set of prime divisors of the order of a finite group G with $\{p_1, \dots, p_n\}$ the set of prime divisors of the order of the normal subgroup N of G. A set of Sylow subgroups $\{P_1, \dots, P_n, P_{n+1}, \dots, P_k\}$, $P_i \in \mathrm{Syl}_{p_i}(G)$, form a *strong Sylow system with respect to* N if $P_i P_j$ is a mutually permutable product for all $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, k\}$. We show that a finite group G is a solvable PST-group if and only if it has a normal subgroup N such that G/N is nilpotent and G has a strong Sylow system with respect to N. It is also shown that G is a solvable PST-group if and only if G has a normal solvable PST-subgroup N and G/N'' is a solvable PST-group.

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1. Introduction

All groups in this paper are finite. A subgroup H of a group G is said to *permute* with the subgroup K if HK is a subgroup of G. H is said to be *permutable* (*S-permutable*) in G if it permutes with all the subgroups (Sylow subgroups) of G. Subgroups H and K of G are said to be *mutually permutable* in G provided that H permutes with every subgroup of K and K permutes with every subgroup of H. Further, H and K are said to be *totally permutable* in G if every subgroup of H permutes with every subgroup of K. If H and K are mutually permutable subgroups of G such that $H \cap K = 1$, then they are totally permutable (see Lemma 1 of [11]). Mutually permutable and totally permutable products have been studied in [2–5,11–13].

A set of pairwise permutable Sylow subgroups of a group G, one for each prime dividing the order of G, is called a *Sylow basis* of G. Let $\{P_1, P_2, \dots, P_r\}$ be a Sylow basis of G. If $\pi = \{p_{i_1}, p_{i_2}, \dots, p_{i_k}\}$ is a set of prime divisors of $|G|$ and P_{i_j} is the Sylow p_{i_j}-subgroup from the given Sylow basis, then $H = P_{i_1} P_{i_2} \cdots P_{i_k}$ is a Hall π-subgroup of G, that is $([G:H], |H|) = 1$. A second Sylow basis $\{Q_1, Q_2, \dots, Q_r\}$ is said to be conjugate to $\{P_1, P_2, \dots, P_r\}$ if there is an element $g \in G$ such that $P_i^g = Q_i$, $1 \leqslant i \leqslant r$. Phillip Hall (see [14,20]) showed that a group is solvable if and only if it has a Sylow basis. He also proved that any two Sylow bases of G are conjugate.

Kegel [17] proved that an S-permutable subgroup is subnormal. We say that S-permutability is a transitive relation in a group G if a subgroup H which is S-permutable in an S-permutable subgroup K of G is S-permutable in G. A group G is called a *PST-group* if S-permutability is a transitive relation in G. By Kegel's result PST-groups are exactly those groups in which all subnormal subgroups are S-permutable. Likewise permutability is a transitive relation in G if every subnormal subgroup of G is permutable. Groups in which permutability is a transitive relation are called PT-groups. Examples of PT-groups include the *Iwasawa* groups, the groups all of whose subgroups are permutable. A group G is called a T-group if normality is a transitive relation in G. *Dedekind* groups, that is, groups all of whose subgroups are normal, are examples of T-groups. These classes have been investigated in [1,7–10,15,19,22]. We note that the class of T-groups is a proper subclass of the class of PT-groups which is a proper subclass of the class of PST-groups. Also, every nilpotent group is a PST-group. The classes of solvable T (resp. PT and PST)-groups are closed under forming subgroups and quotients.

Agrawal [1] showed that a group G is a solvable PST-group if and only if it has an abelian normal Hall subgroup N on which G acts by conjugation as power automorphisms and G/N is a solvable PST-group. G is a solvable T (resp. PT)-group if and only if it is a solvable PST-group whose Sylow subgroups are Dedekind (resp. Iwasawa). The subgroup N mentioned above may be taken to be the nilpotent residual of G (see [1,15,22]).

The purpose of this paper is to provide a number of new characterizations of solvable PST (resp. T and PT)-groups. These characterizations will be given in Theorems A–C.

Let $\{P_1, P_2, \dots, P_k\}$ be a Sylow basis for the solvable group G. Then $G = P_1 P_2 \cdots P_k$ is the product of the pairwise permutable Sylow subgroups $P_1, P_2, \dots, P_k$. If the subgroups $P_1, P_2, \dots, P_k$ are mutually permutable in pairs, then the set $\{P_1, P_2, \dots, P_k\}$ is called a *strong Sylow basis* for G. Note that in a strong Sylow basis the elements are in fact pairwise totally permutable (see Lemma 1 of [11]). By Theorem 16 of [2], a group with a strong Sylow basis is supersolvable.

Let $\{p_1, p_2, \dots, p_k\}$ be the set of prime divisors of $|G|$ and let L be a normal subgroup of G. Let us call $\Sigma = \{P_1, P_2, \dots, P_k\}$ a *strong Sylow system of G with respect to L* if Σ consists of Sylow subgroups of G, $P_i \in \mathrm{Syl}_{p_i}(G)$, $1 \leqslant i \leqslant k$, where for each $P_i \in \Sigma$ with $p_i \in \pi(L)$ we have that $P_i P_j$ is a mutually permutable product for all $P_j \in \Sigma$. Note that it is not inherently clear from the definition that a group G with a strong Sylow system Σ with respect to a normal subgroup L is solvable, for if p_m and p_n are primes not in $\pi(L)$ with corresponding Sylow subgroups P_m and P_n in Σ it is not clear that $P_m P_n = P_n P_m$. Nevertheless, we arrive at the following theorem.

Theorem A. *Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N. The following statements are equivalent*:

(a) *G has a strong Sylow basis,*
(b) *G has a strong Sylow system with respect to N,*
(c) *G is a solvable PST-group.*

The next result is a consequence of Theorem A and the results mentioned earlier.

Corollary 1. *Let G be a group and N be a normal subgroup of G with nilpotent quotient group G/N. The following statements are equivalent*:

(a) *G has a strong Sylow basis and the Sylow subgroups of G are Iwasawa* (*Dedekind*),
(b) *G has a strong Sylow system with respect to N and the Sylow subgroups of G are Iwasawa* (*Dedekind*),
(c) *G is a solvable PT-group* (*T-group*).

By entirely different methods Qinhai [18] proved the equivalence of (a) and (c) in Theorem A and Corollary 1.

Another somewhat different characterization of solvable PST-groups is the following.

Theorem B. *A group G is a solvable PST-group if and only if it has a normal solvable PST-subgroup N such that G/N'' is a solvable PST-group.*

Since solvable PST-groups can be characterized in terms of their Sylow bases, it only seems natural that there is a way to characterize them in terms of their system normalizers. In [6] Ballester-Bolinches, Cossey and Soler-Escriva prove the following rather surprising and beautiful theorem.

Theorem 1. *Let G be a solvable group. If the subgroup H of G permutes with all the system normalizers of G, then H is subnormal in G.*

Our next result provides a characterization of solvable PST-groups in terms of their system normalizers.

Theorem C. *Let G be solvable with D a system normalizer of G.*

(a) *G is a PST-group if and only if D is a Hall subgroup with Hall complement L for which every subgroup of L permutes with D.*
(b) *G is a PT-group if and only if D is an Iwasawa Hall subgroup with Hall complement L for which every subgroup of L permutes with D.*
(c) *G is a T-group if and only if D is a Dedekind Hall subgroup with Hall complement L for which every subgroup of L permutes with D.*

The following corollary is immediate.

Corollary 2. *Let G be a solvable group with D a system normalizer of G.*

(a) *G is a PST-group if and only if D is a Hall subgroup with Hall complement L where $G = LD$ is a mutually permutable product of L with D.*
(b) *G is a PT-group if and only if D is an Iwasawa Hall subgroup with Hall complement L where $G = LD$ is a mutually permutable product of L with D.*
(c) *G is a T-group if and only if D is a Dedekind Hall subgroup with Hall complement L where $G = LD$ is a mutually permutable product of L with D.*

2. Proofs of the main results

We begin with three lemmas which are used in the proofs of Theorems A–C.

Lemma 1. *Let G be a group with normal subgroup N.*

(a) *If $\{P_1, P_2, \ldots, P_k\}$ is a strong Sylow basis of G, then $\{P_1N/N, P_2N/N, \ldots, P_kN/N\}$ is a strong Sylow basis of G/N.*
(b) *If $\{P_1, P_2, \ldots, P_k\}$ is a strong Sylow system of G with respect to M, then $\{P_1N/N, P_2N/N, \ldots, P_kN/N\}$ is a strong Sylow system of G/N with respect to MN/N.*

Proof. Both results are clear from the definitions and standard arguments. □

Lemma 2. *Let N be a normal subgroup of the group G such that N and G/N'' are supersolvable. Then G is supersolvable. In particular, if N and G/N'' are solvable PST-groups, then G is supersolvable.*

Proof. By Theorem 5.4.10 of [20], N' is nilpotent. Hence G is supersolvable by a result in [21]. The last part of the lemma follows since a solvable PST-group is supersolvable by Theorem 2.2 of [1]. □

Lemma 3. *Let G be a solvable PST-group, F the Fitting subgroup of G, L the nilpotent residual of G and $Z_*(G)$ the hypercenter of G. Then L is an abelian Hall subgroup of G and $F = L \times Z_*(G)$.*

Proof. We have L is an abelian Hall subgroup of G by Theorem 2.3 of [1]. That $F = L \times Z_*(G)$ follows from the supersolvability of G. □

Proof of Theorem A. A strong Sylow basis of the group G is also a strong Sylow system with respect to any normal subgroup N of G. So (a) implies (b).

Next we prove that (b) implies (c). Let N be a normal subgroup of G such that G/N is nilpotent and G has a strong Sylow system with respect to N. Let $\{p_1, p_2, \ldots, p_r\}$ be the set of prime divisors of $|N|$ and let $\Sigma = \{P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_k\}$ be a strong Sylow system with respect to N, $P_i \in \mathrm{Syl}_{p_i}(G)$, $1 \leqslant i \leqslant r$. Let $N^* = P_1P_2 \cdots P_r$. Then N^* is a Hall subgroup of G which is a supersolvable subgroup of G by Theorem 16 of [2]. Note that NN^* is a subgroup of G and so $N \leqslant N^*$. Since G/N is nilpotent, N^* is a normal subgroup of G. Let p be the largest prime divisor of $|N^*|$ and let $P \in \mathrm{Syl}_p(N^*)$. Then P is normal in G. By induction and part (b) of Lemma 1, G/P is a solvable PST-group.

Let q be a prime divisor of $|G|, q \neq p$, and let $Q \in \mathrm{Syl}_q(G) \cap \Sigma$. Then PQ is a totally permutable product in G. Let X be a subgroup of P. Then XQ is a subgroup of G and X is a subnormal Sylow p-subgroup of XQ. Therefore, X is normal in XQ and $O^p(G)$ acts as power automorphisms on P. By the Schur–Zassenhaus Theorem, G has a Hall p'-subgroup H such that G is a semidirect product of P by H and H acts as power automorphisms on P.

By a result of Huppert [16], P is abelian or $[H, P] = 1$. If P is abelian, then G is a solvable PST-group by Theorem 2.5 of [1]. Otherwise, $G = H \times P$ which means G is a solvable PST-group by Proposition 1.5 of [1]. Thus (c) is a consequence of (b).

To show that (a) follows from (c) we assume that G is a counterexample with a minimal number of prime divisors. Surely G is not a p-group. Let q be the largest prime divisor of $|G|$. Then the Sylow q-subgroup Q of G is normal and a complement H to Q satisfies (c) and (a) since $G/Q = HQ/Q \cong H/(H \cap Q) = H$. If $\{S_1, S_2, \ldots, S_r\}$ is a strong Sylow basis of H, then $\{S_1, S_2, \ldots, S_r, Q\}$ is a strong Sylow basis of G since all subgroups of Q are S-permutable and so QS_i is a mutually permutable product for all i. Now G satisfies (a) and no counterexample exists. This completes the proof of Theorem A. □

Proof of Theorem B. Assume that G has a normal solvable PST-subgroup N such that G/N'' is a solvable PST-group. We are to show that G is a solvable PST-group and we proceed by induction on $|G|$. Since the class of solvable PST-groups is subgroup and quotient closed, it follows that the proper homomorphic images of G satisfy the assumptions on G listed above.

By Lemma 2 G is a supersolvable group. Let q be the largest prime divisor of $|G|$. Then G has a normal Sylow q-subgroup, say Q.

We first assume $Q \cap N'' = 1$. By hypothesis, G/N'' is a PST-group and by Lemma 3 it is q-nilpotent or QN''/N'' is contained in the nilpotent residual of G/N''. If G/N'' is q-nilpotent, the same is true for G, and $G = H \times Q$ with Hall q'-subgroup H of G. Since $H \cong G/Q$ is a PST-group, so is G. We have by Lemma 3 that QN''/N'' is contained in the nilpotent residual of G/N'', and hence QN''/N'' is abelian and G/N'' induces only power automorphisms on QN''/N'' by Theorem 2.3 of [1]. Now $QN''/N'' \cong Q/Q \cap N'' = Q$ and this is an operator automorphism, so G operates on Q only by power automorphisms. Again, G/Q is a PST-group, and since Q is a Sylow subgroup of G, the same is true for G by Theorem 2.4 of [1].

Finally, assume $Q \cap N'' \neq 1$. Then the Sylow q-subgroup $Q \cap N$ of the PST-group N has nontrivial intersection with the hypercenter of N. Therefore $Q \cap N$ is not contained in the nilpotent residual of N by Lemma 3. We have therefore that N is q-nilpotent and $Q \cap N'' = (Q \cap N)''$; $Q \cap N/Q \cap N'' \cong (Q \cap N)N''/N''$ is nonabelian. This means that also QN''/N'' is nonabelian and G/N'' is q-nilpotent by Lemma 3 and Theorem 2.1 of [1]. Consider a Sylow p-subgroup P of G with $p \neq q$. Then Q is a nilpotent normal subgroup of PQ and $[P, Q] \subseteq Q \cap N'' = (Q \cap N)'' \subseteq Q'$. By the famous result of Phillip Hall (see [21]), we obtain that PQ is nilpotent. Now $G = H \times Q$ with Hall q'-subgroup H of G. Again $H \cong G/Q$ and G are PST-groups. The converse follows since the class of solvable PST-groups is subgroup and quotient closed. This completes the proof. □

Proof of Theorem C. Let G be a solvable PST-group, L the nilpotent residual of G and D a system normalizer of G. By Theorem 2.3 of [1], L is an abelian Hall subgroup on which G acts by conjugation as group of power automorphisms. D complements L by Theorem 9.2.7 of [20] and is therefore also a Hall subgroup of G.

Suppose then that $G = LD$ is a product of a system normalizer D with L where L is a Hall complement to D in G. Further suppose every subgroup of L permutes with D. Let $g \in G$. Then $LD^g = G$ so that L permutes with all system normalizers of G. Hence L is subnormal in G by Theorem 1. But then L is a subnormal Hall subgroup of G and hence normal in G.

Now let H be any subgroup of L. If $g \in G$, then since L is normal in G and every subgroup of L permutes with D, $H^{g^{-1}}D \leqslant G$. Hence $HD^g \leqslant G$. So H permutes with all system normalizers of G and must be subnormal in G by Theorem 1. Hence all subgroups of L are subnormal in G. Thus D acts as power automorphisms on L and, in particular, L is nilpotent. Since L is normal in G and every subgroup of D permutes with L, $G = LD$ is a mutually permutable product. Since $L \cap D = 1$, we have that $G = LD$ is in fact a totally permutable product by Lemma 1 of [11]. Since both L and D are nilpotent, Theorem 1 of [5] says that the nilpotent residual of G is abelian. Note $G/L \cong DL/L$ is nilpotent and so L contains the nilpotent residual of G. By a result of Gaschütz, Schenkman and Carter (see Theorem 9.2.7 of [20]) the nilpotent residual of G is complemented by D. Hence L is the nilpotent residual of G. Thus G acts as power automorphisms on L and by Theorem 2.5 of [1] G is a PST-group.

Parts (b) and (c) follow from part (a) and Theorems 3.1 and 3.2 of [1]. □

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