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A note on finite $\mathscr{PST}$-groups

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Abstract. A finite group G is said to be a $\mathscr{PST}$-group if, for subgroups H and K of G with H Sylow-permutable in K and K Sylow-permutable in G, it is always the case that H is Sylow-permutable in G. A group G is a $\mathscr{T}^*$-group if, for subgroups H and K of G with H normal in K and K normal in G, it is always the case that H is Sylow-permutable in G. In this paper, we show that the classes of finite $\mathscr{PST}$-groups and finite $\mathscr{T}^*$-groups coincide. A new characterization of soluble $\mathscr{PST}$-groups is also presented.

1 Introduction and statement of results

Throughout this paper, all groups considered are finite. A subgroup H of a group G is called *Sylow-permutable* in G, or *S-permutable*, if $HS = SH$ for every Sylow subgroup S of G. Kegel [9] has shown that S-permutable subgroups are subnormal. However there exist subnormal subgroups which are not S-permutable. Robinson [10] called $\mathscr{PST}$-groups the groups in which every subnormal subgroup is S-permutable. From Kegel's result, a group G is a $\mathscr{PST}$-group if and only if S-permutability is a transitive relation in G.

Many papers have studied $\mathscr{PST}$-groups in detail. Agrawal initiated the study in [1] where he characterized the soluble $\mathscr{PST}$-groups as follows:

Theorem 1. *A group G is a soluble $\mathscr{PST}$-group if and only the nilpotent residual D of G is an abelian Hall subgroup of odd order such that G induces power automorphisms in D.*

Robinson, in [10], gave the following characterization of $\mathscr{PST}$-groups:

Theorem 2. *A group G is a $\mathscr{PST}$-group if and only if it has a perfect normal subgroup D such that*

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(1) *G/D is a soluble $\mathscr{PST}$-group;*

(2) *$D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$ where $U_i/Z(D)$ is simple and $U_i \trianglelefteq G$;*

(3) *if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$, where $1 \leqslant r < k$, then the factor group $G/U'_{i_1}U'_{i_2}\ldots U'_{i_r}$ satisfies N_p for all $p \in \pi(Z(D))$.*

Here, a group G satisfies N_p if, for all soluble normal subgroups N, the p'-elements of G induce power automorphisms in $O_p(G/N)$.

On the other hand, Asaad and Csörgő defined in [4] *$\mathscr{T}^*$-groups* as the groups G such that if H is a normal subgroup of K and K is a normal subgroup of G, then H is S-permutable in G. In other words, a group G is a $\mathscr{T}^*$-group whenever every subnormal subgroup of G of defect at most 2 is S-permutable in G. The proofs of most results of [4] seem to use the requirement that all subnormal subgroups of a $\mathscr{T}^*$-group are S-permutable, as in $\mathscr{PST}$-groups, without explicitly stating the equivalence between the concepts of a $\mathscr{T}^*$-group and a $\mathscr{PST}$-group. Therefore, in order to check the validity of the proofs of [4], it is necessary to know whether $\mathscr{PST}$-groups can be characterized as the groups in which every subnormal subgroup of defect at most 2 is S-permutable. Our first main result shows that this question has an affirmative answer.

Robinson established in [10] that $\mathscr{PST}$-groups are $\mathscr{SC}$-groups, that is, groups whose chief factors are all simple. With small changes to Robinson's proof of that result, one can arrive at the same conclusion for $\mathscr{T}^*$-groups.

Lemma 3. *A $\mathscr{T}^*$-group is an $\mathscr{SC}$-group.*

$\mathscr{SC}$-groups are also characterized by Robinson in [10].

Theorem 4. *A group G is an $\mathscr{SC}$-group if and only if there is a perfect normal subgroup D such that G/D is supersoluble, $D/Z(D)$ is a direct product of G-invariant simple groups, and $Z(D)$ is supersolubly embedded in G (i.e., there is a G-admissible series of $Z(D)$ with cyclic factors).*

A $\mathscr{U}_p^*$-group is defined in [2] to be a p-supersoluble group G in which all p-chief factors are G-isomorphic when regarded as modules over G. In [2, Corollary 3], the following characterization of soluble $\mathscr{PST}$-groups is given.

Theorem 5. *A group G is a soluble $\mathscr{PST}$-group if and only if it satisfies $\mathscr{U}_p^*$ for all primes p.*

Our first main result shows that $\mathscr{PST} = \mathscr{T}^*$:

Theorem A. *A group G is a $\mathscr{T}^*$-group if and only if it is a $\mathscr{PST}$-group.*

In [3, Theorem 3.1], Asaad proved that a group G is a soluble $\mathscr{T}$-group if and only if for all primes p dividing the order of $F^*(G)$, the generalized Fitting subgroup of G, every p-subgroup of G is pronormal in G. As a consequence, he proved that a group G is a soluble $\mathscr{T}$-group if and only if for all primes p dividing the order of $F^*(G)$, G satisfies property $\mathscr{C}_p$, that is, every subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$ ([3, Corollary 3.2]). He extended this result to permutability by showing that a group G is a soluble $\mathscr{PT}$-group if and only if G satisfies $\mathscr{X}_p$ for all primes p dividing the order of $F^*(G)$. Here, a group G satisfies $\mathscr{X}_p$ when every subgroup of a Sylow p-subgroup P of G is permutable in $N_G(P)$. This property was introduced and studied in [7].

The $\mathscr{PST}$-version of the properties $\mathscr{C}_p$ and $\mathscr{X}_p$ is the property $\mathscr{Y}_p$ introduced in [5]. Recall that a group G satisfies $\mathscr{Y}_p$ if whenever H and K are p-subgroups of G such that $H \leqslant K$, then H is S-permutable in $N_G(K)$. In [5, Theorem 4], it is proved that a group G is a soluble $\mathscr{PST}$-group if and only if G satisfies $\mathscr{Y}_p$ for all primes p. Asaad's results admit the following generalization to $\mathscr{PST}$-groups:

Theorem B. *A group G is a soluble $\mathscr{PST}$-group if and only if G satisfies $\mathscr{Y}_p$ for all primes p dividing the order of* $F^*(G)$.

Unlike previous characterizations of soluble $\mathscr{PST}$-groups, this one does not follow quickly from the classification of minimal non-$\mathscr{PST}$-groups given by Robinson in [11].

2 Proofs

Proof of Theorem A. Only the necessity of the condition is in doubt. We assume that it does not hold and derive a contradiction. Let G be a group of minimal order such that G is a $\mathscr{T}^*$-group but G is not a $\mathscr{PST}$-group. An argument similar to the one used in [1] to show that quotients of $\mathscr{PST}$-groups are $\mathscr{PST}$-groups shows that all quotient groups of G are $\mathscr{T}^*$-groups. Therefore, by minimality of G, every proper quotient group of G is a $\mathscr{PST}$-group. Applying Lemma 3, G is an $\mathscr{SC}$-group. Thus, from Theorem 4, we have that G has a normal perfect subgroup D such that $D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$, with all $U_i/Z(D)$ simple, and $Z(D)$ is supersolubly embedded in G.

Assume that $D \neq 1$, i.e., G is not soluble. Then G/D is a soluble $\mathscr{PST}$-group. Since $U_i/Z(D)$ is simple for all i, we have $U_i' \neq 1$ for all i. Therefore if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ with $r < k$, then G/U_{i_j}' is a $\mathscr{PST}$-group and so $G/U_{i_1}'U_{i_2}' \ldots U_{i_r}'$ satisfies N_p for all primes p. Theorem 2 implies that G is a $\mathscr{PST}$-group, contrary to assumption. Therefore $D = 1$ and G is soluble. Since all chief factors of G are simple, G is supersoluble. Let p be the largest prime dividing the order of G. Then G has a normal Sylow p-subgroup, P say. Moreover, G/P is a $\mathscr{PST}$-group by the choice of G. Hence G/P satisfies $\mathscr{U}_q^*$ for all primes $q \neq p$ by Theorem 5. This implies that G satisfies $\mathscr{U}_q^*$ for all primes $q \neq p$. Since G is not a $\mathscr{PST}$-group, it follows that G does not satisfy $\mathscr{U}_p^*$.

Suppose that $O_{p'}(G) \neq 1$. Then $G/O_{p'}(G)$ is a soluble $\mathscr{PST}$-group. Therefore $G/O_{p'}(G)$ satisfies $\mathscr{U}_p^*$ by Theorem 5, and so G satisfies $\mathscr{U}_p^*$. This is a contradiction.

Consequently $O_{p'}(G) = 1$. Assume that G has two different minimal normal subgroups N_1 and N_2. Both of them have order p, and G/N_1 and G/N_2 satisfy $\mathscr{U}_p^*$. If $N_1 N_2$ is a proper subgroup of P, then by considering all chief factors of G between N_1 and $N_1 N_2$, between N_2 and $N_1 N_2$, and between $N_1 N_2$ and P, we obtain that G satisfies $\mathscr{U}_p^*$. This contradiction shows that $P = N_1 N_2$. Note that $P = N_1 \times N_2$ is abelian. If D is a subgroup of P, then D is normal in P and D is S-permutable in G; hence D is normalized by all p'-elements of G and D is normal in G. Thus elements of G induce power automorphisms in P, from which it follows that G satisfies $\mathscr{U}_p^*$, contrary to the choice of G.

Hence G has a unique minimal normal subgroup N, which is contained in P, and G/N is a $\mathscr{PST}$-group. Moreover, $P = O_p(G) = F(G)$, the Fitting subgroup. If N is not contained in the Frattini subgroup $\Phi(G)$ of G, then G is a primitive group and so $N = F(G)$ has order p. In particular, G satisfies $\mathscr{U}_p^*$. This contradiction yields that $N \leqslant \Phi(G)$. If G/N is p-nilpotent, then G is p-nilpotent and so G satisfies $\mathscr{U}_p^*$. This is not possible. Consequently, G/N is not nilpotent. Since G/N is a $\mathscr{PST}$-group, the nilpotent residual R/N of G/N is an abelian Hall subgroup of G/N and all elements of G induce power automorphisms on R/N. Moreover, N is the unique minimal normal subgroup of G. In particular, R/N is a p-group and so $P/N = R/N$. In particular, p'-elements of G/N induce power automorphisms on P/N. Let S be a subgroup of P. Then SN is normal in G because P/N is abelian and $O^p(G/N)$ normalizes SN/N. In addition, since either $SN = S$ or S is a maximal subgroup of the p-group SN, we have that S is a normal subgroup of SN. Since G is a $\mathscr{T}^*$-group, S is S-permutable in G. Then all p'-elements of G normalize S and so induce power automorphisms in P. Hence G satisfies $\mathscr{U}_p^*$. This is the final contradiction.

The following lemma is needed in the proof of Theorem B.

Lemma 6. *Let p be a prime and let M be a normal p'-subgroup of a group G. Then G satisfies $\mathscr{Y}_p$ if and only if G/M satisfies $\mathscr{Y}_p$.*

Proof. By [5, Lemma 2], if G satisfies $\mathscr{Y}_p$, then G/M satisfies $\mathscr{Y}_p$. Conversely, assume that G/M satisfies $\mathscr{Y}_p$. By [5, Theorem 5], either G/M is p-nilpotent, or G/M has abelian Sylow p-subgroups and G/M satisfies $\mathscr{C}_p$. In the first case, G is p-nilpotent and so G satisfies $\mathscr{Y}_p$ by [5, Theorem 5]. Assume that G/M has abelian Sylow p-subgroups and satisfies $\mathscr{C}_p$. Let P be a Sylow p-subgroup of G. Consider a subgroup H of P, and $g \in N_G(P)$. We have $H^g M = HM$ because HM/M is normalized by $gM \in G/M$. Therefore

$$H^g = H^g M \cap P = HM \cap P = H.$$

This implies that G satisfies $\mathscr{C}_p$ and so G satisfies $\mathscr{Y}_p$ by [5, Theorem 5].

Proof of Theorem B. If G is a soluble $\mathscr{PST}$-group, we can apply [5, Theorem 4] to conclude that G satisfies $\mathscr{Y}_p$ for all primes p. Let G be a group satisfying $\mathscr{Y}_p$ for all primes p dividing the order of $F^*(G)$. We shall prove that G is a soluble $\mathscr{PST}$-group by induction on $|G|$. By [5, Theorem 4], we can suppose that $F^*(G)$ is a proper subgroup of G. Note that the class $\mathscr{Y}_p$ is subgroup-closed for all primes p. Hence $F^*(G)$ satisfies $\mathscr{Y}_p$ for all primes p. Applying [5, Theorem 4], we have that $F^*(G)$ is soluble. Therefore $1 \neq F^*(G) = F(G)$ by [8, (X, 13)].

Suppose that there exists a prime p dividing $|F^*(G)|$ such that a Sylow p-subgroup P of G is not abelian. In this case, G is p-nilpotent by [5, Theorem 5]. Moreover, since $F^*(O_{p'}(G))$ is contained in $F^*(G)$, we have that $O_{p'}(G)$ is a soluble $\mathscr{PST}$-group by induction. This implies that G is soluble. Let N be a minimal normal subgroup of G contained in $O_p(G)$. Since $N \cap Z(P)$ is non-trivial and contained in the centre of G, we have $N \cap Z(P) = N$. Thus $F(G/N) = F(G)/N$. Consequently G/N satisfies $\mathscr{Y}_p$ for all primes p dividing $|F^*(G/N)|$. Hence G/N is a soluble $\mathscr{PST}$-group by induction and so G/N satisfies $\mathscr{Y}_q$ for all primes q dividing $|G/N|$ by [5, Theorem 4]. By Lemma 6, G satisfies $\mathscr{Y}_q$ for all primes $q \neq p$. Since G satisfies $\mathscr{Y}_p$ by hypothesis, it follows that G satisfies $\mathscr{Y}_p$ for all primes p and so G is a soluble $\mathscr{PST}$-group by [5, Theorem 4].

Therefore we can assume, by [5, Theorem 5], that for every prime p dividing $|F^*(G)|$, G has an abelian Sylow p-subgroup P and G satisfies $\mathscr{C}_p$. In this case, every cyclic subgroup of p-power order of $F(G)$ is normal in G, because G satisfies $\mathscr{C}_p$, and so centralized by G'. Hence G' is contained in $C_G(F(G))$, which is contained in $F(G)$ by [8, (X, 13)]. Thus G' is abelian and so G is soluble.

Let q be a prime. If q divides $|G'|$, then q divides $|F(G)|$ and so G satisfies $\mathscr{Y}_q$ by hypothesis. Suppose that q does not divide $|G'|$. Consider a q-subgroup H of G. We have that HG' is a normal subgroup of G and so every Sylow subgroup of HG' is pronormal in G. Hence H is pronormal in G. According to [6, Lemma 2], G satisfies $\mathscr{C}_q$ and so G satisfies $\mathscr{Y}_q$ by [5, Theorem 3]. Consequently, G is a soluble $\mathscr{PST}$-group.

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