Corrigendum: A note of finite $\mathcal{PST}$-groups

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In Robinson [4, Theorem 3.1], there is a very subtle typo; the inequality in condition (iii) should read $0 \leq r < k$ with the convention that when $r = 0$ the corresponding empty product is taken to be the trivial subgroup. So, statement (3) of Theorem 2 in our article should be changed accordingly. Unfortunately, we used Theorem 2 exactly as it was stated in our proof of Theorem A. Hence to complete the proof of Theorem A, we need to show G satisfies N_p for every $p \in \pi\bigl(\mathrm{Z}(D)\bigr)$ taking care of the case when $r = 0$. First we need the following lemma.

Lemma 1. *Let N be a normal p-subgroup of a group G, p a prime. Then the p'-elements of G induce power automorphisms on N if and only if all chief factors of G below N are cyclic and G-isomorphic.*

Proof. Assume that all p'-elements of G induce power automorphisms on N. Then every chief factor of G below N is centralized by all p-elements of G. If a p'-element g of G does not centralize N, then N is abelian by a theorem of Huppert [2, Hilfssatz 5]. Therefore the power automorphism induced by such an element g is universal by a result of Cooper ([3, 13.4.3]). It follows that all chief factors of G below N are cyclic and G-isomorphic. If all p'-elements g of G centralize N, then all chief factors of G below N are central.

Assume now that all chief factors of G below N are cyclic and G-isomorphic. Let q be a prime different from p and let Q be a Sylow q-subgroup of G. Then $X = NQ$ is a subgroup of G and all p-chief factors of X are cyclic and X-isomorphic. Therefore X satisfies $\mathcal{U}_p^*$. Since every subgroup H of N is subnormal and p'-perfect, it follows that H permutes with Q by [1, Theorem 6]. On the other hand, H is normal in HQ. Thus Q normalizes H. Consequently, all p'-elements of G act as power automorphisms on N. □

Let now assume we are in the context of the proof of Theorem A. In particular, G is a group of minimal order such that G is a $\mathcal{T}^*$-group but not a $\mathcal{PST}$-group, G is an $\mathcal{SC}$-group and all proper factor groups of G are $\mathcal{PST}$-groups. So, to complete the proof of our theorem in the insoluble case, it is enough to check that G satisfies N_p for every $p \in \pi\big(\mathrm{Z}(D)\big)$. By the remark in postscript (ii) of [4], we have that G is a $\mathcal{PST}$-group if and only if its non-abelian chief factors are simple and each quotient of G satisfies N_p. Since the proper factor groups of G are $\mathcal{PST}$-groups, it is enough to show that for each $p \in \pi\big(\mathrm{Z}(D)\big)$, the p'-elements of G induce power automorphisms in $\mathrm{O}_p(G)$. The result is clear if $|\mathrm{O}_p(G)| \leq p$. Assume $|\mathrm{O}_p(G)| > p$. Let N be a minimal normal subgroup of G contained in $\mathrm{O}_p(G)$. Then all p'-elements of G/N induce power automorphisms in $\mathrm{O}_p(G/N) = \mathrm{O}_p(G)/N$. Lemma 1 implies that all chief factors of G/N below $\mathrm{O}_p(G)/N$ are cyclic and G-isomorphic. Consider now a minimal normal subgroup A/N of G/N contained in $\mathrm{O}_p(G)/N$. Since G is an $\mathcal{SC}$-group, it follows that A is an abelian normal subgroup of G of order p^2. If H is a subgroup of A, then H is a subnormal subgroup of G of defect at most 2. Since G is a $\mathcal{T}^*$-group, H is S-permutable in G. It follows that every subgroup of A is normalized by all Sylow q-subgroups of G with $q \neq p$. In particular, the p'-elements of G act as power automorphisms on A. By Lemma 1, all chief factors of G below A are cyclic and G-isomorphic. Consequently all chief factors of G below $\mathrm{O}_p(G)$ are cyclic and G-isomorphic and so the p'-elements of G induce power automorphisms on $\mathrm{O}_p(G)$ by Lemma 1.

References

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