Groups with a Finite Covering by Isomorphic Abelian Subgroups

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Dedicated to Dr. James C. Beidleman on the occasion of his seventieth birthday.

Abstract. In this paper, we look at groups with a finite covering by proper isomorphic abelian subgroups (CIA-groups). Our main focus will be on finite groups with such a covering. In particular, we will see that there are no simple CIA-groups, but on the other hand, that every finite group is a direct factor of a CIA-group. A complete characterization of finite abelian CIA-groups is given and, modulo their Sylow structure, a characterization of finite nilpotent CIA-groups is given. We also show that a CIA-group G must contain an element whose order is the exponent of G. It is trivial that groups where the exponent of the group and the exponent of the center coincide satisfy this property. Hence we consider the class of CIA-groups G whose exponent is greater than the exponent of $Z(G)$. We say such groups have a "small" center. A question we leave open is whether or not there exist centerless CIA-groups. Such a group must have a "small" center providing us with further motivation to study CIA-groups with small center. Lastly, the role of GAP in this work is discussed.

1. Introduction

A group is said to have a covering by subgroups if it is the set-theoretic union of proper subgroups, and, if the set of subgroups is finite, we say the covering is finite. To develop our theme, let us briefly look at the background and history of group coverings. Results on finite coverings by subgroups first appeared in a book by Scorza [**19**] with an emphasis on coverings by a small number of subgroups. Bernhard Neumann in [**15**] and [**16**] investigated coverings by cosets. The following theorem, often called Neumann's Lemma, is a key to many group theoretic results. In particular, a characterization of groups having finite coverings is stated as a corollary of the following theorem.

Theorem 1.1. *Let $G = \bigcup_{i=1}^{k} g_iH_i$, where $H_1, \dots, H_k$ are (not necessarily distinct) subgroups of G. Then, if we omit from the union any cosets g_iH_i for which $[G : H_i]$ is infinite, the union of the remaining cosets is still all of G.*

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COROLLARY 1.2. *A group has a finite covering by subgroups if and only if it has a finite non-cyclic homomorphic image.*

The following unpublished result by Reinhold Baer (see Theorem 4.6 in [**17**]) leads to the investigation of finite coverings by special subgroups as can be found in [**5**] and [**13**].

THEOREM 1.3. *A group is central-by-finite if and only if it is the union of finitely many abelian subgroups.*

Coverings have been widely studied in groups, and recently, analogous coverings for rings, semigroups, and loops have been discussed in [**1**], [**14**], and [**7**], respectively. In the first author's study of loops covered by subgroups (see [**7**], [**8**], and [**9**]), a family of loops that are covered by isomorphic abelian subgroups was encountered.

DEFINITION 1.4. Given $(\mathbb{F}, +, \cdot)$ a field, and a finite idempotent quasigroup $(Q, \odot)$, let $\mathcal{L}^{(Q)}(\mathbb{F}) = \{a_q(x) : x \in \mathbb{F}^* \text{ and } q \in Q\} \cup \{\mathbf{1}\}$ (i.e. each element of the form $a_q(x)$ in this set is double indexed by q and x) be the loop whose binary operation is defined as follows:

i. For any $l \in \mathcal{L}^{(Q)}(\mathbb{F})$, $\mathbf{1}l = l\mathbf{1} = l$.
ii. For $x, y \in \mathbb{F}^*$,

$$a_q(x)a_q(y) = \begin{cases} a_q(x+y) & \text{if } x+y \neq 0 \\ \mathbf{1} & \text{otherwise} \end{cases}$$

iii. For $x, y \in \mathbb{F}^*$, $a_{q_1}(x)a_{q_2}(y) = a_{q_1 \odot q_2}(xy)$ for $q_1 \neq q_2$.

The loops in this family are covered by $|Q|$ copies of $(\mathbb{F}, +)$. These loops can even be simple. In view of this family of loops and Theorem 1.3, it is only natural to ask what one can say about groups which are covered by isomorphic abelian subgroups. This is the primary aim of this work.

In this paper, our main focus will be on finite groups with a finite covering by proper isomorphic abelian subgroups. We call such groups CIA-groups. In Section 2, we give some basic motivating examples and show that direct products of CIA-groups are CIA-groups. Section 3 is devoted to a complete characterization of finite abelian CIA-groups and a characterization of finite nilpotent CIA-groups modulo their Sylow structure. In particular, we will see that a group G is a finite abelian CIA-group if and only if G has a direct factor which is a direct product of two isomorphic cyclic subgroups. The finite nilpotent CIA-groups will be seen to be those finite nilpotent groups whose Sylow subgroups are abelian or CIA-groups with at least one Sylow subgroup being a CIA-group. We will see in Section 4 that the groups of square-free order, the dihedral groups (excluding the Klein four group), the Frobenius groups, the symmetric groups, the alternating groups, and all simple groups are examples of non-CIA-groups. However, in Theorem 5.1, it is shown that every finite group is a direct factor of a CIA-group. This result gives rise to several examples of CIA-groups.

In Section 5, we give examples of finite CIA-groups which have "small" centers, that is, CIA-groups where the exponent of the group is greater than the exponent of its center. This section asks the question, "When is the center of a CIA-group 'small'?" Let us give some motivation for this question. As one will see, every example we give of a CIA-group will have a nontrivial center. We have been unable to prove that this is always the case but if a centerless CIA-group exists, then it certainly has a small center. This was our initial motivation for searching for group with a small center. In Theorem 4.1, we show that a finite CIA-group G must contain an element whose order is the exponent of G. One way to guarantee that a finite group has such an element is to force the exponent of the group and its center to be the same. CIA-groups with a small center will certainly satisfy Theorem 4.1, but, in some sense, in a nontrivial fashion. Lastly, we will see that groups constructed using Theorem 5.1 will not have a small center. So CIA-groups with a small center will need to be constructed in some other manner. How one can construct such CIA-groups with a small center takes up the remainder of Section 5.

It should be pointed out that this work would not have been possible without the use of GAP [**10**]. In Section 6, we give the details of our GAP documentation, but let us briefly elaborate on the elementary, yet crucial, role GAP played in this research. Without knowing much about CIA-groups, we looked to examples to help us understand how CIA-groups arise and to understand their structure. GAP was used to determine which groups in GAP's "SmallGroups" [**2**] library were CIA-groups. A function called `IsUnionOfIsomorphicSubgroups` was defined for GAP which we used to check which groups in the "SmallGroups" library were CIA-groups. The GAP command `StructureDescription` was then used to give us a decomposition of the CIA-groups found into various direct and semidirect products of common groups. Nearly all of the results in the above paragraphs were first conjectured by looking at pages of GAP output consisting of different CIA-groups and their structure descriptions as given by GAP.

2. Preliminaries

In this section, we will look at some basic motivating examples. We will also prove a basic lemma concerning direct products of groups covered by isomorphic abelian subgroups.

Throughout, we will denote the cyclic group of order n by C_n. Also, $\exp(G)$ will be used to denote the exponent of the group G.

Definition 2.1. A group G has a finite covering by proper isomorphic abelian subgroups if $G = \bigcup_{i=1}^{n} A_i$ where the A_i's are proper isomorphic abelian subgroups of G. We will call such groups CIA-groups.

Example 2.2. Let G be a finite group. If $\exp(G) = p$ where p is a prime and G is not cyclic, then G is a CIA-group since G is covered by all subgroups of order p.

Example 2.3. Let $G = A_4 \times C_3$ where A_4 denotes the alternating group on 4 elements. One can check that G is not a CIA-group. Note that G contains the Klein four group K_4 as a normal subgroup. However K_4 and $G/K_4 \simeq C_3 \times C_3$ are both CIA-groups. So the class of CIA-groups is not closed with respect to forming extensions.

Example 2.4. The abelian group $C_4 \times C_2$ is not a CIA-group.

Example 2.5. The quaternion group Q_8 is a CIA-group, since Q_8 is covered by all subgroups of order 4.

Example 2.6. Let G be a finite group. If $\exp(G) = p^2$ where p is a prime, then $G \times C_{p^2}$ is a CIA-group, since $G \times C_{p^2}$ is covered by all subgroups isomorphic to $C_{p^2} \times C_p$.

If G is a nonabelian group of $\exp(G) = p^2$ and $|G| = p^3$ where p is an odd prime, then G is not a CIA-group since there are elements of order p that do not commute with any element of order p^2. So one can easily see that CIA-groups are not closed with respect to taking subgroups or quotients.

However, it is the case that CIA-groups are closed with respect to taking direct products. Also worth noting is the fact that the direct product of a CIA-group with an abelian group is again a CIA-group. This tells us that every abelian group is a direct factor of a CIA-group. We will see in Section 5 that this is true of any finite group.

LEMMA 2.7. *Let H be a CIA-group. Suppose K is either a CIA-group or an abelian group. Then $H \times K$ is a CIA-group.*

PROOF. Suppose first that K is a CIA-group. Write $H = \bigcup_{i=1}^{h} H_i$ and $K = \bigcup_{j=1}^{k} K_j$ each as unions of isomorphic abelian groups. Then

$$H \times K = \bigcup_{\substack{1 \leq i \leq h \\ 1 \leq j \leq k}} H_i \times K_j$$

is a covering of $H \times K$ by isomorphic abelian groups.

Simply replace each K_j above with K and one can easily see that $H \times K$ is a CIA-group when K is abelian. □

3. Abelian and Nilpotent *CIA*-groups

In this section, we will examine the structure of finite abelian CIA-groups and finite nilpotent CIA-groups. Throughout, superscripts on groups are used for indexation only. We begin with a definition that will be important in characterizing the finite abelian CIA-groups which are of prime power order.

DEFINITION 3.1. Given a finite group G, $g \in G$ is a *maximum root of* G provided whenever $h^n = g$ we have $|g| = |h|$. Let $\pi_{me}(G)$ be used to denote the following:

$$\pi_{me}(G) = \{|g| : g \text{ is a maximum root in } G\}.$$

First let us examine the homocyclic abelian p-groups. These groups are CIA-groups provided they have more than one component.

LEMMA 3.2. *Let $P = P_1 \times P_2 \times \cdots \times P_t$ be homocyclic where each P_i is a cyclic group of order p^α. Then P is the union of subgroups isomorphic to C_{p^α}. Moreover, if $t > 1$, then P is a CIA-group.*

PROOF. Let $x \in P$. We will show $x \in H \simeq C_{p^\alpha}$ for some $H \leq P$. Suppose each P_i is generated by the element z_i. Then $x = z_1^{\gamma_1} \cdots z_t^{\gamma_t}$ where $0 \leq \gamma_i < p^\alpha$ for all i. We can suppose p divides γ_i for all i or else $|x| = p^\alpha$ in which case we let $H = \langle x \rangle$. Let p^δ be the largest power of p for which p^δ divides γ_i for all i. Then $z_1^{\gamma_1/p^\delta} \cdots z_t^{\gamma_t/p^\delta}$ has order p^α. Now $x \in H = \langle z_1^{\gamma_1/p^\delta} \cdots z_t^{\gamma_t/p^\delta} \rangle \simeq C_{p^\alpha}$ and the desired result follows. □

We are now in a position to characterize finite abelian CIA-groups of prime power order.

Theorem 3.3. *Let P be an abelian p-group with order p^β. Using the Frobenius-Stickelberger Theorem, write $P = P_1 \times \cdots \times P_t$ with $P_i = C^1_{p^{\beta_i}} \times \cdots \times C^{n_i}_{p^{\beta_i}}$ where $C^k_{p^{\beta_i}} \simeq C_{p^{\beta_i}}$ for all k and i, $\beta_i \neq \beta_j$ for $i \neq j$, and $\sum_{i=1}^t n_i\beta_i = \beta$. Then the following hold:*

(i) *P equals the union of subgroups isomorphic to $C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$;*
(ii) *P is a CIA-group if and only if $n_i > 1$ for some i.*

Proof. (i) If $n_i = 1$ for all i, then $P \simeq C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$ and the result is trivial. If $t = 1$, then P is homocyclic and Lemma 3.2 gives the desired result. So we can suppose $n_i \neq 1$ for some i and $t \neq 1$. Without loss of generality, let us suppose $n_1 \neq 1$. Let $\overline{P}_1 = C^2_{p^{\beta_1}} \times \cdots \times C^{n_1}_{p^{\beta_1}}$ and let $\overline{P} = \overline{P}_1 \times P_2 \times \cdots \times P_t$. Note that $P = C^1_{p^{\beta_1}} \times \overline{P}$.

Let $x \in P$. We will show x is an element of some subgroup of P isomorphic to $C = C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$. By induction on $|P|$, we have that $\overline{P}$ can be written as the union of subgroups of $\overline{P}$ isomorphic to C. So if $x \in \overline{P}$, then x is in some subgroup of $\overline{P}$, and hence a subgroup of P, isomorphic to C. Clearly if $x \in C^1_{p^{\beta_1}}$, then x is in some subgroup of P isomorphic to C. So we can suppose that $x = zy$ where z and y are nontrivial elements of $C^1_{p^{\beta_1}}$ and $\overline{P}$, respectively. Write $y = y_1y_2$ with $y_1 \in \overline{P}_1$ and $y_2 \in P_2 \times \cdots \times P_t$. By induction, y_2 is in some subgroup of $P_2 \times \cdots \times P_t$ isomorphic to $C_{p^{\beta_2}} \times \cdots \times C_{p^{\beta_t}}$. By Lemma 3.2, zy_1 is in some subgroup of P_1 isomorphic to $C_{p^{\beta_1}}$. Hence $x = zy_1y_2$ is an element of some subgroup of P isomorphic to $C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$.

(ii) If $n_i > 1$, then (i) says P is a CIA-group.

Suppose $n_i = 1$ for all i. Then, we may write $P = C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$. Supposing P is a CIA-group, write $P = \bigcup_{i=1}^m A_i$ where each $A_i \simeq C_{p^{\alpha_1}} \times \cdots \times C_{p^{\alpha_t}}$ with $1 \leq p^{\alpha_i} \leq p^{\beta_i}$ for all i and $p^{\alpha_j} < p^{\beta_j}$ for at least one j. Let $\langle x \rangle = C_{p^{\beta_j}}$ and note x is a maximum root of P. Now, for some i, we have $x \in A_i$. By the structure of A_i, x must be equal to some power of an element, say y, of A_i, where y has order larger than that of x. This contradicts the fact that x is a maximum root of P. Thus P is not a CIA-group. □

It should come as no surprise that the structure of a finite abelian CIA-group depends on the group's Sylow structure. Here we see that a finite abelian group is a CIA-group if and only if it possesses a Sylow CIA-subgroup.

Theorem 3.4. *Let G be a finite abelian group with order $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ where each p_i is prime and $p_i \neq p_j$ for $i \neq j$. Then G is a CIA-group if and only if some Sylow subgroup of G is a CIA-group.*

Proof. Suppose some Sylow subgroup, say P, of G is a CIA-group. Let H be a Sylow p-complement to P in G. Then $G = P \times H$ and we see G is a CIA-group after applying Lemma 2.7.

Suppose no Sylow subgroup of G is a CIA-group. Applying part (ii) of Theorem 3.3, we see that for each $P_i \in \mathrm{Syl}_{p_i}(G)$,

$$P_i = C_{p_i^{\beta_{i_1}}} \times \cdots \times C_{p_i^{\beta_{i_{t(i)}}}}$$

where $t(i)$ is the number of factors in P_i, $\beta_{i_k} \neq \beta_{i_j}$ for $k \neq j$, and $\sum_{k=1}^{t(i)} \beta_{i_k} = \alpha_i$. Assume G is a CIA-group. Then G is the union of proper subgroups isomorphic to $A_1 \times \cdots \times A_n$ where for each A_i,

$$A_i \simeq C_{p_i^{\gamma_{i_1}}} \times \cdots \times C_{p_i^{\gamma_{i_{t(i)}}}}$$

with $1 \leq p_i^{\gamma_{i_j}} \leq p_i^{\beta_{i_j}}$ for all i and j. Also, we must have $p_l^{\gamma_{l_m}} < p_l^{\beta_{l_m}}$ for at least one l and m. Let x be an element generating $C_{p_l^{\beta_{l_m}}}$. Then x is a maximum root of P_l. There must exist a subgroup H of G isomorphic to $A_1 \times \cdots \times A_n$ containing x, and x must be an element of the Sylow p_l-subgroup of H, say H_l, which is isomorphic to A_l. By the structure of H_l, x must be equal to some power of an element, say y, of H_l, where y has order larger than that of x. This contradicts the fact that x is a maximum root of P_l. Thus G is not a CIA-group. □

COROLLARY 3.5. *A finite abelian group G is a CIA-group if and only if there exists a direct factor of G of the form $C_{p^n} \times C_{p^n}$ for some prime p and some integer n.*

NOTATION 3.6. For a set S of integers, D_S will be used to denote the direct product $D_S = \bigtimes_{n \in S} C_n$.

COROLLARY 3.7. *Let G be a finite abelian group. Then G is a CIA-group if and only if $P \not\simeq D_{\pi_{me}(P)}$ for some Sylow subgroup P of G.*

COROLLARY 3.8. *A finite cyclic group is not a CIA-group.*

COROLLARY 3.9. *Let G be a finite abelian CIA-group with $\{A_i\}_{i=1}^n$ a collection of isomorphic abelian subgroups of G. Then $\{A_i\}_{i=1}^n$ is a covering of G if and only if $A_i \simeq \left(\bigtimes_{P \in \mathrm{Syl}(G)} D_{\pi_{me}(P)}\right) \times H$ where H is any subgroup of G trivially intersecting and not complementing $\left(\bigtimes_{P \in \mathrm{Syl}(G)} D_{\pi_{me}(P)}\right)$ in G.*

We will end this section with two results on nilpotent CIA-groups. The next theorem will tell us that, ultimately, to characterize the nilpotent CIA-groups, one needs to characterize the CIA-groups of prime power order.

THEOREM 3.10. *Let G be a finite nilpotent group. Write $G = P_1 \times \cdots \times P_n$ where $P_i \in \mathrm{Syl}_{p_i}(G)$. Then G is a CIA-group if and only if G satisfies the following two conditions:*

(1) *Every Sylow subgroup of G is abelian or is a CIA-group;*
(2) *At least one Sylow subgroup of G is a CIA-group.*

PROOF. Suppose every Sylow subgroup of G is abelian or is a CIA-group and further suppose G possesses at least one Sylow subgroup which is a CIA-group. Then we can decompose G into $G = H \times K$, where H is abelian and K is the direct product of CIA-groups. We observe that K is a CIA-group by Lemma 2.7 and then, after applying Lemma 2.7 once more, we see that G is a CIA-group.

Suppose G is a CIA-group. Write $G = \bigcup_{i=1}^m A^i$ where the A^i's are isomorphic abelian groups. Since G is nilpotent, we can write each A^i as $A^i = A_1^i \times \cdots \times A_n^i$ where each $A_j^i \leq P_j$. Let $x \in P_j$. Then $x \in A^i$ for some i and, since x is of p_j-power order, it follows that $x \in A_j^i$. Hence $P_j = \bigcup_{i=1}^m A_j^i$. If A_j^i is proper in P_j, then P_j is a CIA-group. If $A_j^i = P_j$, then P_j is abelian. Thus (1) holds.

To show (2) holds, we can suppose P_j is abelian for all j. Then G is an abelian CIA-group and hence possesses a Sylow subgroup which is a CIA-group by Theorem 3.4. □

COROLLARY 3.11. *Every finite Hamiltonian group is a CIA-group.*

Proof. This follows from Theorem 3.10, Lemma 2.7, and the fact that the quaternion group of order 8 is a CIA-group, □

4. Non-CIA-groups

In this section we will give several examples of groups which are not CIA-groups. First, we will look at a theorem and a corollary which will aid in showing certain groups are not CIA-groups.

Theorem 4.1. *Let G be a finite CIA-group and write $G = \bigcup_{i=1}^{n} H_i$, where the H_i's are proper isomorphic abelian groups. Then, given any $x \in G$, there exists an $a \in G$ with $|a| = \exp(G)$ and $a \in C_{H_i}(x)$ for some i. In particular, $\exp(G) = \exp(H_i)$ for all i.*

Proof. Given any $x \in G$, then x is in at least one H_i. So $|x|$ divides $|H_i|$ for all i and thus $\exp(G)$ divides $|H_i|$ for all i. So for each H_i there is an $a_i \in H_i$ with $|a_i| = \exp(G)$. Given any $x \in G$, then x is in at least one H_i and $a_i \in C_{H_i}(x)$. □

Corollary 4.2. *Let G be a CIA-group. If $x \in G$ and p is a prime divisor of $|G|$, then p is a divisor of $|C_G(x)|$. Moreover, $\exp(G)$ is a divisor of $|C_G(x)|$.*

The converse of Theorem 4.1 does not hold. Let $G = C_4 \times C_2$. Then G is not a CIA-group, but certainly if $x \in G$ then $\exp(G)$ divides $|C_G(x)|$, since G is abelian.

Now we will look at several classes of groups which are not CIA-groups.

Proposition 4.3. *There are no CIA-groups of square-free order.*

Proof. Assume that G is CIA-group of square-free order. Let $G = \bigcup_{i=1}^{m} A_i$ with the A_i's isomorphic abelian subgroups of G. By Theorem 4.1, $\exp(G) = \exp(A_i)$. But G is of square-free order and so $|G| = \exp(G)$ giving us $G = A_i$. Hence G is not a CIA-group. □

Proposition 4.4. *If $G = D_n$, the dihedral group of order $2n$, where $n > 2$, then G is not a CIA-group.*

Proof. First note that D_4 is not a CIA-group. Let x and y be the generators of D_n, where $|x| = 2$ and $|y| = n \geq 3$. Then $|C_{D_n}(x)| \leq 4 \leq \exp(D_n)$. Supposing G is a CIA-group, we see from Theorem 4.1 that $|C_{D_n}(x)|$ is n or $2n$. If $|C_{D_n}(x)| = 2n$, then x is central in D_n which is not the case. So $|C_{D_n}(x)| = n$. We can assume $\exp(G) = n$ as well. Hence $n = 4$ and we have a contradiction. So D_n is not a CIA-group. □

Proposition 4.5. *If G is a Frobenius group, then G is not a CIA-group.*

Proof. Assume G is a Frobenius CIA-group. Write $G = HN$ with $H \cap N = 1$ where N is the Frobenius kernel of G. Let p be a prime dividing $|H|$ and let x be a nontrivial element of N. By Exercise 8.5.5 of [**18**], we have that p does not divide $|N|$. By Exercise 8.5.6 of [**18**], we have that $C_G(x) \leq N$. However, by Corollary 4.2, we have that p divides $|C_G(x)|$ in which case p divides $|N|$. Hence there are no Frobenius groups which are CIA-groups. □

Proposition 4.6. *If $G = S_n$, the symmetric group of degree n, where $n \geq 2$, then G is not a CIA-group.*

PROOF. By Corollary 3.8, S_2 is not a CIA-group. Suppose $n > 2$. If n is odd, let x be an n-cycle so that $|C_{S_n}(x)| = n$. If n is even, let x be an $(n-1)$-cycle so that $|C_{S_n}(x)| = n - 1$. It is now evident that S_n is not a CIA-group. □

PROPOSITION 4.7. *If $G = A_n$ the alternating group of degree n where $n \geq 3$, then G is not a CIA-group.*

PROOF. By Corollary 3.8, A_3 is not a CIA-group. Suppose $n > 3$. If n is odd, let x be an n-cycle so that $|C_{A_n}(x)| = n$. If n is even, let x be an $(n-1)$-cycle so that $|C_{A_n}(x)| = n - 1$. It is now evident that A_n is not a CIA-group. □

NOTATION 4.8. For a finite group G, let us agree to denote by $\pi_e(G)$ the set of all orders of elements in G, that is

$$\pi_e(G) = \{|a| : a \in G\}.$$

PROPOSITION 4.9. *If G is a simple group, then G is not a CIA-group.*

PROOF. By Theorem 1.3, there are no infinite simple CIA-groups, since CIA-groups are central-by-finite. Assume G is a finite simple CIA-group. Note that by Theorem 4.1, we have $\exp(G) \in \pi_e(G)$. Then G must be abelian by Theorem 6 of [**20**]. Hence G is cyclic and Corollary 3.8 gives us a contradiction showing us that there are no simple CIA-groups. □

5. *CIA*-groups

All the groups that we will look at in this section are finite. We now have several examples of groups which are not CIA-groups. However, the list of CIA-groups is still large, for in this section we will see that every group is a direct factor of a CIA-group. We we will end the section with several examples of CIA-groups (mostly found with the use of GAP [**10**]) and concern ourselves with the question, "When is the center of a CIA-group 'small'?'

THEOREM 5.1. *If G is a finite group, then $G \times D_{\pi_e(G)}$ is a CIA-group covered by abelian subgroups isomorphic to $D_{\pi_e(G)}$.*

PROOF. Let $H = G \times D_{\pi_e(G)}$ and $\Pi_1 : H \to G$ be the projection homomorphism. Given $h \in H$, let $g = \Pi_1(h)$, $k = |g|$, Π_t be the projection homomorphism from H to the k^{th}-order factor of $D_{\pi_e(G)}$, $l = \Pi_t(h)$, and $A = D_{\pi_e(G)-\{k\}} \times \langle gl \rangle$. Then $h \in A$ and $A \simeq D_{\pi_e(G)}$. □

COROLLARY 5.2. *Every finite group is a direct factor of a CIA-group.*

COROLLARY 5.3. *If G is a finite group and $\exp(G) = \prod_{n \in \pi_e(G)} n$, then G is a direct factor of a CIA-group which is covered by copies of the cyclic group $C_{\exp(G)}$.*

PROOF. $D_{\pi_e(G)} = C_{\exp(G)}$. □

EXAMPLE 5.4. Let A_5 be the alternating group on 5 elements and let $G = A_5 \times C_{30}$. Using Theorem 5.1 we see that G is a non-solvable CIA-group and is covered by copies of C_{30}.

For the remainder of this section, we will concern ourselves with the center of a CIA-group. In particular, we are interested in finding examples of CIA-groups with a "small" center.

Definition 5.5. Let us say that a group G has a *small center* if $\exp\big(Z(G)\big) < \exp(G)$.

Remark 5.6. If $\exp(G) > |Z(G)|$, then G has a small center.

Example 5.7. The quaternion group Q_8 is a CIA-group with a small center.

The next example shows that there exist p-groups with a small center, for any odd prime p, which are CIA-groups.

Example 5.8. Let $G = \langle x, y \mid x^{p^2} = y^{p^2} = 1,\ x^y = x^{p+1} \rangle$ with p an odd prime. Then the center of G is $\langle x^p \rangle \times \langle y^p \rangle$ and the exponent of G is p^2. Also, G is a CIA-group with small center and G is covered by copies of C_{p^2}.

Proof. Note that G is a an extension of C_{p^2} by itself and thus has order p^4.

Let us first show G has $\langle x^p \rangle \times \langle y^p \rangle$ as its center. We have $(x^p)^y = (x^y)^p = (x^{p+1})^p = x^{p^2+p} = x^p$ and hence $x^p \in Z(G)$. Note $x^y = x^{p+1}$ says that $[x, y] = x^p$. Hence $[y, x] = x^{-p}$ and one can deduce that $y^x = yx^{-p}$. Thus one has $(y^p)^x = (y^x)^p = (yx^{-p})^p = y^p x^{-p^2} = y^p$ so that $y^p \in Z(G)$. Now $|Z(G)| \neq p^3$ or else $G/Z(G)$ is cyclic in which case G is abelian. So the center of G must be $\langle x^p \rangle \times \langle y^p \rangle$.

Note that $G/Z(G)$ must be elementary abelian of order p^2 so that G has class 2. By Lemma 3.9 in [**11**], we have that $(gh)^p = g^p h^p$ for all g and h in G. In particular, we see that G has exponent p^2.

To show that G is a CIA-group covered by copies of C_{p^2}, we only need to verify that each element $g \in G$ of order p is in a cyclic group of order p^2. Let g be of order p in G. If g is a power of x or a power of y, then we have nothing to show since g would be in $\langle x \rangle$ or $\langle y \rangle$. So write $g = x^i y^j$ with both i and j not divisible by p^2. Note $1 = g^p = x^{ip} y^{jp}$. So p divides both i and j. Write $i = tp$ and $j = kp$ and note that p does not divide t nor k. Hence $h = x^t y^k$ has order p^2 and $g \in \langle h \rangle$. □

The following theorems will allow for some interesting examples of CIA-groups.

Theorem 5.9. *Let H be a group of exponent p for some prime p. Let Q_8 denote the quaternion group of order 8. If $G = HQ_8$ where each element of H commutes with an element of order 4 in Q_8, each element of Q_8 commutes with an element of order p in H, and the center of G has order divisible by 2, then G is a CIA-group covered by copies of C_{4p}.*

Proof. Firstly, note that since the center of G has order divisible by 2, we have that G contains a unique element of order 2 which is found in each conjugate of Q_8. Secondly, note that the hypotheses imply that every element of order p commutes with one of order 4 and that every element of order 4 commutes with one of order p. These facts are easily verified using Sylow's Theorem.

Each element of G is of order 1, 2, 4, p, $2p$, or $4p$. It is clear that the elements of order 1, 2, and $4p$ can be found as elements in certain cyclic groups of order $4p$. The identity and the unique element of order 2 will be an any subgroup of order $4p$ and any element of order $4p$ will be in the group generated by itself.

Let g be an element of order 4 and let h be of order p such that g and h commute. Then gh is of order $4p$ and we have $g \in \langle g^p \rangle = \langle (gh)^p \rangle \leq \langle gh \rangle$.

Let g be an element of order $2p$. Then g^2 has order p and g^p has order 2. Since g^2 has order p, we know g^4 has order p as well. So $g^4 g^p$ must be of order $2p$ and so $g \in \langle g^4 g^p \rangle$. Now, let k be an element of order 4 commuting with g^2. So $g^2 k$

is of order $4p$ and $k^2 = g^p$. We have $g \in \langle g^4 g^p \rangle = \langle g^4 k^2 \rangle = \langle (g^2 k)^2 \rangle \leq \langle g^2 k \rangle$. This completes the proof. $\square$

THEOREM 5.10. *Let H and K be groups such that $\exp(H) = p$ and $\exp(K) = q$ where p and q are distinct primes. If $G = HK$ where every element in K commutes with an element of order p in H and every element in H commutes with an element of order q in K, then G is a CIA-group covered by copies of C_{pq}.*

PROOF. As in the proof of Theorem 5.9, Sylow's Theorem and the hypotheses imply that every element of order p commutes with one of order q and every element of order q commutes with one of order p. Since G is a product of a Sylow p-subgroup of $\exp(H) = p$ and a Sylow q-subgroup of $\exp(K) = q$, every $g \in G$ has order 1, p, q, or pq. Let g be an element of order p and let k be an element of order q such that g and k commute. Then gk has order pq and $g \in \langle g^q \rangle = \langle (gk)^q \rangle \leq \langle gk \rangle$. Likewise, if g is an element of order q commuting with h an element of order p, then $g \in \langle gh \rangle$. $\square$

The next example is a nonabelian CIA-group of odd order having a small center.

EXAMPLE 5.11. Let

$$H = \langle a, b, c \mid a^7 = b^7 = c^7 = 1, a^c = a, b^c = b, a^b = ac \rangle$$

be the nonabelian group of order 7^3 and exponent 7 and let

$$K = \langle d, e, f \mid d^3 = e^3 = f^3 = 1, d^f = d, e^f = e, d^e = df \rangle$$

be the nonabelian group of order 3^3 and exponent 3. Define the group G as a semidirect product $H \rtimes K$ where the action of K on H is defined as follows; e and f centralize H, $a^d = a^2$, $b^d = b$, and $c^d = c^2$. So, in short, we have G is the following:

$$\left\langle a,b,c,d,e,f \;\middle|\; \begin{array}{c} a^7{=}b^7{=}c^7{=}d^3{=}e^3{=}f^3{=}1,\ a^c{=}a,\ b^c{=}b,\ a^b{=}ac, \\ d^f{=}d,\ e^f{=}e,\ d^e{=}df,\ a^e{=}a,\ b^e{=}b,\ c^e{=}c, \\ a^d{=}a^2,\ b^d{=}b,\ c^d{=}c^2,\ a^f{=}a,\ b^f{=}b,\ c^f{=}c \end{array} \right\rangle.$$

It is apparent from the relations that the center of K is $\langle f \rangle$ and, moreover, $\langle f \rangle$ is in the center of G. Note from the relations that the center of H must be $\langle c \rangle$. If the center of G contains a subgroup of order 7 then it must contain $\langle c \rangle$. However c does not commute with d. So the center of G is $\langle f \rangle$ and is of order 3.

Clearly f commutes with any $h \in H$. Also, it is apparent from the relations that b commutes with any $k \in K$. So by Theorem 5.10 we see G is a CIA-group covered by copies of C_{21} with a center of order 3. Thus G is a non-nilpotent CIA-group of odd order with a small center.

EXAMPLE 5.12. Let

$$K = \langle a, b, c \mid a^p = b^p = c^p = 1, a^c = a, b^c = b, a^b = ac \rangle$$

be the nonabelian group of order p^3 and exponent p and let

$$Q_8 = \langle x, y \mid y^4 = 1, x^2 = y^2, x^y = x^{-1} \rangle$$

be the quaternion group of order 8. Let us define two groups, G and H, as semidirect products $K \rtimes Q_8$ where the action of Q_8 on K is defined as follows; $a^x = a$, $b^x = b^{-1}$, and $c^x = c^{-1}$ for G, $a^x = a^{-1}$, $b^x = b^{-1}$, and $c^x = c$ for H, and y centralizes K for both G and H. So, in short, we have the following two groups:

$$G = \left\langle a,b,c,x,y \;\middle|\; \begin{array}{l} a^p{=}b^p{=}c^p{=}x^4{=}y^4{=}1,\ x^2{=}y^2,\ a^c{=}a, \\ b^c{=}b,\ a^b{=}ac,\ x^y{=}x^{-1},\ a^x{=}a,\ b^x{=}b^{-1}, \\ c^x{=}c^{-1},\ a^y{=}a,\ b^y{=}b,\ c^y{=}c \end{array} \right\rangle,$$

$$H = \left\langle a,b,c,x,y \;\middle|\; \begin{array}{l} a^p{=}b^p{=}c^p{=}x^4{=}y^4{=}1,\ x^2{=}y^2,\ a^c{=}a, \\ b^c{=}b,\ a^b{=}ac,\ x^y{=}x^{-1},\ a^x{=}a^{-1},\ b^x{=}b^{-1}, \\ c^x{=}c,\ a^y{=}a,\ b^y{=}b,\ c^y{=}c \end{array} \right\rangle.$$

From the relations used to define both G and H, one sees that a, b, and c each commute with y. Hence each element of K commutes with an element of order 4 in Q_8. Also, a and x commute in G and c and x commute in H. Hence each element of Q_8 commutes with an element of order p in K. Also the centers of G and H have orders divisible by 2, since they contain $\langle x^2 \rangle$. So the hypotheses of Theorem 5.9 are satisfied. Hence we see that both G and H are CIA-groups covered by copies of C_{4p}.

Also, the relations make it clear that $\langle x^2 \rangle \leq Z(G)$ and $\langle x^2, c \rangle \leq Z(H)$. Since Q_8 has no element of 4 in its center, neither does G nor H. Suppose $g \in Z(G)$ and $|g| = p$. Then $g \in K$ and so $g \in Z(K) = \langle c \rangle$. So g is c or a p'-power of c. However, c nor any p'-power of c commutes with x as the relation $c^x = c^{-1}$ shows. Now suppose $g \in Z(H)$ and $|g| = p$. Then $g \in K$ and so $g \in Z(K) = \langle c \rangle$. We can conclude that $Z(G) = \langle x^2 \rangle$ and $Z(H) = \langle x^2, c \rangle$. So, not only are G and H CIA-groups, they are CIA-groups with small centers.

It seems likely that by using Theorem 5.10 and a product of two "large" groups, one of exponent p and another of exponent q with q dividing $p-1$, one will be able to construct a centerless CIA-group. However, we have been unable to construct such a group. It is still an open question as to whether centerless CIA-groups exist.

6. GAP Documentation

Let us briefly elaborate on the elementary, yet crucial, role GAP [**10**] played in this research. Without knowing much about CIA-groups, we looked to examples to help us understand how CIA-groups arise and to understand their structure. GAP was used to determine which groups in GAP's "SmallGroups" library were CIA-groups. The following function was defined in GAP:

```
# Determines if G is the union of its subgroups isomorphic to S.
#
IsUnionOfIsomorphicSubgroups:=
     function(G,S)
          local c;      # All subgroups of G isomorphic to S.
          c:=Filtered(ConjugacyClassesSubgroups(G),
                 C->IdGroup(S)=IdGroup(Representative(C)));
          return IsSubset(Concatenation(
                 List(Concatenation(List(c,Elements)),Elements)),G);
end;
```

This function allows one to determine if a given group `G` can be written as the union of subgroups all isomorphic to `S`. Note that no attempt was made to ensure this was done in the quickest manner possible. We hoped that several small examples would be sufficient for us to gain a better understanding of CIA-groups and hence memory issues would not be a problem. The following GAP function allows one to determine which small groups, in a certain range of orders, are CIA-groups:

```
# Computes CIA groups for all groups whose order
# is between low to high.
#
CIAGroups:=
  function(low,high)
      local i,j,           # index variables
            G,Agrps,       # Group in question and possible abelian
                           # subgroups of G
            A;             # Abelian groups
      for i in [low..high] do
          for j in [1..NumberSmallGroups(i)] do
              G:=SmallGroup(i,j);
              Agrps:=Filtered(AllGroups([1..Order(g)-1]),
                          H->(IsAbelian(H) and
                          exponent(H)=Exponent(G)));
              for A in Agrps do
                  if IsUnionOfIsomorphicSubgroups(G,A) then
                       Print(IdGroup(G)," ",IdGroup(A)," ",
                             StructureDescription(G),
                             " is covered by copies of ",
                             StructureDescription(A),"\n");
                  fi;
              od;
          od;
      od;
end;
```

It should be mentioned that this code will not run if 1024 is between `low` and `high` because groups of this order are excluded from the "SmallGroups" library. In the above, `Agrps` is a list of all groups `H` with order less than the order of `G` where `H` is abelian and exp(`H`)=exp(`G`). The code checks if `G` is a CIA-group by making use of the previously defined function `IsUnionOfIsomorphicSubgroups`. The output makes use of `StructureDescription` which tells us the structure of `G` and the subgroups covering `G`. Not only do wee see which groups are CIA-groups, but we see the different possible coverings used.

The results in Section 3 were discovered by examining the data that GAP provided us. Also, it became apparent that several classes of groups were missing from our lists of CIA-groups. This lead us to the results in Section 4. We also noticed that even though certain groups, like the dihedral, symmetric, and alternating groups, weren't CIA-groups, they kept appearing as direct factors of certain CIA-groups. This lead us to the result in Theorem 5.1. At one point in our research we conjectured that there were no centerless CIA-groups. After all, GAP had not found any. This lead us to search for CIA-groups with small centers. Several groups fitting the hypotheses in Example 5.8 and Theorems 5.9 and 5.10 were found by GAP. After examining these examples, we generalized and discovered the results found in Example 5.8 and Theorems 5.9 and 5.10. All of this was done essentially with the two GAP functions defined in this section.

It should be mentioned that the group G found in Example 5.11 was constructed in GAP as a free group modulo the relations given. We then used GAP to verify that this group was a CIA-group. Of course, Theorem 5.10 does this work for us now.

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