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The intersection map of subgroups and certain classes of finite groups

James C. Beidleman · Matthew F. Ragland

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Abstract The goals of this paper are twofold. One is to look at the behavior of the collections of permutable subgroups and S-permutable subgroups under the intersection map into a fixed subgroup of a group. The other is to locally analyze the intersection map in connection with $\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups. In particular, we generalize Theorem 1 of Bauman [Arch. Math. (Basel) 25:337–340, 1974] to $\mathcal{PT}$- and $\mathcal{PST}$-groups.

Keywords T-Groups · Subnormal · Intersection map · Finite groups · Permutable

Mathematics Subject Classification (2000) 20D10 · 20D30 · 20D35 · 20E15 · 20E25

1 Introduction

All groups considered are finite. All unexplained notation and terminology is standard and can be found in [24]. A group G is a $\mathcal{T}$-group if normality is transitive; that is, if $H \trianglelefteq K \trianglelefteq G$, then $H \trianglelefteq G$. A subgroup H of G is said to be permutable in G if $HK = KH$ for all subgroups K of G. Ore [21] proved that a permutable subgroup

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J. C. Beidleman (✉)
Department of Mathematics, University of Kentucky, 935 Patterson Office Tower,
Lexington, KY 40506-0027, USA
e-mail: clark@ms.uky.edu

M. F. Ragland
Department of Mathematics, Auburn University Montgomery, P.O. Box 244023,
Montgomery, AL, 36124-4023, USA
e-mail: mragland@mail.aum.edu

is subnormal. A group G is a $\mathcal{PT}$-group if permutability is transitive; that is, if H is permutable in K and K is permutable in G, then H is permutable in G. By Ore's result, G is a $\mathcal{PT}$-group if and only if every subnormal subgroup of G is permutable in G. A subgroup H of G is called Sylow-permutable in G, or S-permutable, if H permutes with every Sylow subgroup of G. Kegel [20] showed that an S-permutable subgroup is subnormal. A group G is said to be a $\mathcal{PST}$-group if S-permutability is a transitive relation in G. Applying Kegel's result, a group G is a $\mathcal{PST}$-group if and only if every subnormal subgroup of G is S-permutable in G.

The basic structures of solvable $\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups were established by Gaschütz [17], Zacher [33], and Agrawal [1], respectively, and are presented in the following theorem.

Theorem 1 *Let L be the nilpotent residual of a group G. Then*

(a) (Agrawal [1]) *G is a solvable $\mathcal{PST}$-group if and only if L is an abelian Hall subgroup of G on which G acts by conjugation as a group of power automorphisms.*
(b) (Zacher [33]) *G is a solvable $\mathcal{PT}$-group if and only if G is a solvable $\mathcal{PST}$-group and G/L is an Iwasawa group.*
(c) (Gaschütz [17]) *G is a solvable $\mathcal{T}$-group if and only if G is a solvable $\mathcal{PST}$-group and G/L is a Dedekind group.*

A group G is called an Iwasawa group if every subgroup of G is permutable in G. If every subgroup of G is normal in G, then G is called a Dedekind group. The classification of Iwasawa groups can be found in [29], whereas, the classification of Dedekind groups can be found in [24].

The subject of several papers, in particular, [7], [13], and [23], respectively, has been that of local characterizations of $\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups. In particular, consider the following definitions.

Definition 1 Let G be a group and p a prime. Then

(a) G is a $\mathcal{Y}_p$-group if, for all p-subgroups H and K of G such that $H \leq K$, H is S-permutable in $N_G(K)$.
(b) G is an $\mathcal{X}_p$-group if each subgroup of a Sylow p-subgroup P of G is permutable in $N_G(P)$.
(c) G is a $\mathcal{C}_p$-group if each subgroup of a Sylow p-subgroup P of G is normal in $N_G(P)$.

Each part of the following theorem is, respectively, the culmination of [7], [13], and [23].

Theorem 2 *Let G be a group.*

(a) *G is a solvable $\mathcal{PST}$-group if and only if G is a $\mathcal{Y}_p$-group for all primes p.*
(b) *G is a solvable $\mathcal{PT}$-group if and only if G is an $\mathcal{X}_p$-group for all primes p.*
(c) *G is a solvable $\mathcal{T}$-group if and only if G is a $\mathcal{C}_p$-group for all primes p.*

We note that the class of $\mathcal{T}$-groups forms a proper subclass of the class of $\mathcal{PT}$-groups which in turn forms a proper subclass of the class of $\mathcal{PST}$-groups. These classes have been studied in detail and further information can be found in [1–15,17,22,23,25–27,33].

The behavior of certain families of subgroups under the intersection map into a fixed subgroup of a group has been investigated in a number of different ways by Bauman [12], Venzke [30], Huppert [19], Wielandt [31], and Wielandt and Huppert [32]. Bauman [12] used the intersection map to study solvable $\mathcal{T}$-groups. His approach is considerably different from the approaches used in [8,17,23]. Consider the following definition.

Definition 2 A subgroup H of a group G is said to be *normal sensitive* in G if the map $N \to H \cap N$ sends the lattice of normal subgroups of G onto the lattice of normal subgroups of H, that is, if $\{L|L \trianglelefteq H\} = \{H \cap N|N \trianglelefteq G\}$.

Bauman ties the concept of normal sensitivity to that of $\mathcal{T}$-groups in the following theorem.

Theorem 3 (Bauman [12]) *Every subgroup of G is normal sensitive in G if and only if G is a solvable $\mathcal{T}$-group.*

The goals of this paper are twofold. One is to look at the behavior of the collections of permutable subgroups and S-permutable subgroups under the intersection map into a fixed subgroup of a group. The other is to locally analyze the intersection map in connection with $\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups.

2 The intersection map and the classes $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$

First let us define the analogues of normal sensitivity for permutability and S-permutability.

Definition 3 A subgroup H of a group G is said to be

(a) *permutable sensitive* in G if the following holds:

$$\{N|N \text{ is permutable in } H\} = \{H \cap W|W \text{ is permutable in } G\}.$$

(b) *S-permutable sensitive* in G if the following holds:

$$\{N|N \text{ is S-permutable in } H\} = \{H \cap W|W \text{ is S-permutable in } G\}.$$

The collection of S-permutable subgroups of a group G is a sublattice of the lattice of a subnormal subgroups of G (see [20] and Corollary 1 of [28]) so that a subgroup H of G is S-permutable sensitive if the map $W \to H \cap W$ sends the lattice of S-permutable subgroups W of G onto the lattice of S-permutable subgroups $H \cap W$ of H. Although the collection of permutable subgroups of a group G is a subset of the lattice of subnormal subgroups of G, they need not be a sublattice, as the example (found through the use of GAP [16]) below illustrates. Thus the intersection map, in the case of permutable subgroups in G, need not be a lattice mapping.

Example 1 Define the group G as follows:

$$G = \langle x, y, z | x^8 = y^2 = z^2 = 1, x^y = x, x^z = xy, y^z = y \rangle.$$

So $G \simeq (\langle x \rangle \times \langle y \rangle) \rtimes \langle z \rangle$. Let $A = \langle y, z \rangle$ and $B = \langle yx^4, z \rangle$. It can be shown that $A \trianglelefteq G$ and that B is permutable in G. However, $\langle z \rangle = A \cap B$ is not permutable in G.

One purpose here is to establish the following result which is the analogue of Theorem 3 for solvable $\mathcal{PT}$- and $\mathcal{PST}$-groups.

Theorem A *Let G be a group.*

(a) *G is a solvable $\mathcal{PST}$-group if and only if every subgroup of G is S-permutable sensitive in G.*
(b) *G is a solvable $\mathcal{PT}$-group if and only if every subgroup of G is permutable sensitive in G.*

The proof of Theorem A is much different from the proofs of parts (a) and (b) of Theorem 1. The transitivity of these properties is highly incorporated into the proof of Theorem A.

The assumption in Theorem A that every subgroup of G is S-permutable or permutable sensitive is needed to guarantee the solvability of the group G. Similarly, Theorem 3 uses the assumption of every subgroup being normal sensitive to force solvability. However, if we restrict the S-permutable, permutable, or normal sensitivity to the subnormal subgroups of G, then we can still deduce that G is a $\mathcal{PST}$-, $\mathcal{PT}$-, or $\mathcal{T}$-group, respectively. In fact, for $\mathcal{PST}$-groups and $\mathcal{T}$-groups, we can even restrict S-permutable and normal sensitivty, respectively, to the normal subgroups. This is the content of our next theorem.

Theorem B *Let G be a group. Then*

(a) *G is a $\mathcal{PST}$-group if and only if every normal subgroup of G is S-permutable sensitive in G.*
(b) *G is a $\mathcal{PT}$-group if and only if every subnormal subgroup of G is permutable sensitive in G.*
(c) *G is a $\mathcal{T}$-group if and only if every normal subgroup of G is normal sensitive in G.*

Whether or not one can replace "subnormal" with "normal" in part (b) of Theorem B is an open question.

We now turn to some local considerations for the concepts of normal, permutable, and S-permutable sensitivity. Consider the following definitions.

Definition 4 Let G be a group and p a prime. Then

(a) G is a $\mathscr{Y}_p^*$-group if, for each p-subgroup K of G, each subgroup H of K is S-permutable sensitive in $N_G(K)$.
(b) G is an $\mathscr{X}_p^*$-group if each subgroup of a Sylow p-subgroup P of G is permutable sensitive in $N_G(P)$.

(c) G is a $\mathscr{C}_p^*$-group if each subgroup of a Sylow p-subgroup P of G is normal sensitive in $N_G(P)$.

We establish the following result which localizes the different sensitivity concepts in $\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups.

Theorem C *Let G be a group.*

(a) *G is a solvable $\mathcal{PST}$-group if and only if G is a $\mathscr{Y}_p^*$-group for all primes p.*
(b) *G is a solvable $\mathcal{PT}$-group if and only if G is an $\mathscr{X}_p^*$-group for all primes p.*
(c) *G is a solvable $\mathcal{T}$-group if and only if G is a $\mathscr{C}_p^*$-group for all primes p.*

Theorem C is a consequence of Theorem 2 and the following result.

Theorem D *Let p be a prime. Then the following hold.*

(a) $\mathscr{Y}_p = \mathscr{Y}_p^*$.
(b) $\mathscr{X}_p = \mathscr{X}_p^*$.
(c) $\mathscr{C}_p = \mathscr{C}_p^*$.

3 Preliminary results

In this section we present certain facts and results that are needed to prove Theorems A–D.

The following lemma (see [20] or Corollary 1 of [28]) is a well known result of Kegel's.

Lemma 1 *Let G be a group. If H and K are S-permutable in G, then $H \cap K$ is S-permutable in G.*

Lemma 2 *Let L be the nilpotent residual of the solvable group G and let H be any subgroup of G.*

(a) *If G is a $\mathcal{PST}$-group, then LH is S-permutable in G.*
(b) *If G is a $\mathcal{PT}$-group, then LH is permutable in G.*

Proof Let G be a $\mathcal{PT}$-group ($\mathcal{PST}$-group). Using Theorem 1, we see G/L is an Iwasawa group (nilpotent group). Thus LH/L is permutable (S-permutable) in G/L. Hence LH is permutable (S-permutable) in G. □

It is worth mentioning that if G is a $\mathcal{T}$-group with nilpotent residual L, then $LH \trianglelefteq G$ for any subgroup H of G. This result is not needed in our work, however, it is used in the proof of Theorem 3.

Definition 5 A group G is called an SC-group if all its chief factors are simple.

SC-groups were introduced and classified by Robinson [25].

Lemma 3 *Let G be a group such that every normal subgroup of G is permutable sensitive in G. Then G is an SC-group.*

Proof Let M be a minimal normal subgroup of G. Then every normal subgroup of G/M is permutable sensitive in G/M. By induction, G/M is an SC-group, and thus it is enough to show M is a simple chief factor.

First assume that M is an elementary abelian p-group for some prime p and let P be a Sylow p-subgroup of G. Let x be a nonidentity element of $M \cap Z(P)$. Then $\langle x\rangle$ is a permutable subgroup of M so that there is a permutable subgroup Y of G such that $Y \cap M = \langle x\rangle$. By Lemma 1, $\langle x\rangle$ is S-permutable in G. Let Q be a Sylow q-subgroup of G, $q \neq p$. Then $\langle x\rangle Q = Q\langle x\rangle$ and $\langle x\rangle$ is a subnormal Sylow p-subgroup of $Q\langle x\rangle$. Thus $\langle x\rangle$ is normal in $Q\langle x\rangle$ and $O^p(G)$ normalizes $\langle x\rangle$. Hence $\langle x\rangle$ is normal in G and we deduce M is simple.

Now assume that M is nonabelian. Then $M = M_1 \times M_2 \times \cdots \times M_t$ where each M_i is a nonabelian simple group. Since M_1 is permutable in M there is a permutable subgroup Y of G such that $Y \cap M = M_1$. By Lemma 1, M_1 is S-permutable in G. Let R be a Sylow r-subgroup of G for some prime r. Then $M_1R = RM_1$ and $M_1^R = M_1(M_1^R \cap R)$. Thus M_1^R/M_1 is an r-group. But M_1^R is a direct product of nonabelian simple groups and so $M_1^R = M_1$. Thus R normalizes M_1 and we deduce M_1 is normal in G. So $M = M_1$ the proof is complete. □

The following concepts and results of Robinson [25] are needed to prove part (b) of Theorem B.

Lemma 4 *Let D be the solvable residual of a group G. G is an SC-group if and only if G/D is supersolvable, $D/Z(D)$ is a direct product of G-invariant simple groups, and $Z(D)$ is supersolvably embedded in G (that is, there is a G-admissible series in $Z(D)$ with cyclic factors).*

Definition 6 Let p be a prime.

(a) A group G satisfies condition N_p if, for all solvable normal subgroups N of G, the p'-elements of G induce power automorphisms on $O_p(G/N)$.
(b) A group G satisfies condition P_p if, for all solvable normal subgroups N of G, each subgroup of $O_p(G/N)$ is permutable in a Sylow p-subgroup of G/N.

Theorem 4 *Let D be the solvable residual of a group G. G is a $\mathcal{PT}$-group if and only if*

(a) *G/D is a solvable $\mathcal{PT}$-group;*
(b) *$D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$ where U_i is normal in G and $U_i/Z(D)$ is simple;*
(c) *if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ where $1 \leq r \leq k$, then $G/U'_{i_1}U'_{i_2}\cdots U'_{i_r}$ satisfies N_p for all $p \in \pi\big(Z(D)\big)$ and P_p for all $p \in \pi(D)$.*

4 Proofs of the main results

Proof of Theorem A(b) Suppose every subgroup of G is permutable sensitive in G. Further, suppose G is minimal with respect to not being a $\mathcal{PT}$-group. Let $K \leq H \leq G$.

Then K is permutable sensitive in G. So if L is permutable in K then $L = K \cap N$ where N is permutable in G. Now $N \cap H$ is permutable in H and $L = K \cap N = K \cap H \cap N$. So K is permutable sensitive in H. Thus, by minimality, we can assume every proper subgroup of G is a solvable $\mathcal{PT}$-group. Hence every proper subgroup of G is supersolvable. By a result of Huppert (10.3.4 of [24]), G must be solvable.

Let $N \trianglelefteq G$ with K/N permutable in $H/N \leq G/N$. Then K is permutable in H and so $K = L \cap H$ with L permutable in G. But then $K/N = L \cap H/N$ with L/N permutable in G/N. So we can assume every proper factor group of G is a $\mathcal{PT}$-group.

Suppose G is a p-group. Since G is not a $\mathcal{PT}$-group, G is not an Iwasawa group. By Lemma 2.3.3 of [29], G possesses a section H/K isomorphic to either the dihedral group of order eight or, for p odd, to the nonabelian p-group of order p^3 with exponent p. Since all proper subgroups and all proper factor groups of G are $\mathcal{PT}$-groups, we must have $H = G$ and $K = 1$. It is a straight forward argument to show that the dihedral group of order 8 and all nonabelian p-groups of order p^3 with exponent p do not satisfy the hypothesis.

So, we can now apply Theorem 2 of [26] and deduce that $G = P \rtimes Q$ with Q a cyclic q-group and $P \trianglelefteq G$ where P is either an abelian p-group, $p \neq q$, or P is the quaternion group of order 8, $2 \neq q$.

Let A be subnormal in G with A not permutable in G. Then A is core free since proper quotients of G are $\mathcal{PT}$-groups. First suppose that A is a q-group. Then A is a subnormal Sylow q-subgroup of PA. Hence P normalizes A yielding $A \trianglelefteq G$. So we can assume p divides $|A|$. Let A_p be a Sylow p-subgroup of A. A_p is normal in P and so there is a subgroup T permutable in G with $P \cap T = A_p$. Note that A_p must be a Sylow p-subgroup of T. So $T = A_p T_q$ with T_q a Sylow q-subgroup of T. By choosing conjugates and renaming Q, we can assume $T_q \leq Q$. Now, T permutable in G yields $QT = TQ = A_p Q$. Hence Q normalizes A_p and thus $A_p \trianglelefteq G$. Thus G/A_p is a $\mathcal{PT}$-group so that A/A_p is permutable in G/A_p. We can deduce A is permutable in G and this gives a final contradiction.

Conversely, assume G is a solvable $\mathcal{PT}$-group and let L be the nilpotent residual of G. By Theorem 1, L is a normal abelian Hall subgroup of G. Let C be a system normalizer of G. By a result of Gaschütz, Schenkmen, and Carter (9.2.7 of [24]), $G = LC$ and $C \cap L = 1$. Note that all the complements of L in G are system normalizers of G and hence are conjugate in G. By a result of Carter (9.5.10 of [24]), all the complements to L in G are necessarily Carter subgroups of G. Also notice that C is a Hall subgroup of G.

Let us show every subgroup H of G is permutable sensitive in G. Let T be a permutable subgroup of H. Since factor groups of $\mathcal{PT}$-groups are again $\mathcal{PT}$-groups, we can assume T is core free. Using Lemma 2, we have TL is permutable in G. Assume that $TL \neq G$. By Theorem 1, TL is a solvable $\mathcal{PT}$-group. Note that T is permutable in $H \cap TL$. By induction, there exists K permutable in TL such that $H \cap TL \cap K = T$ so that $H \cap K = T$. But K is permutable in TL and TL is permutable in G which implies K is permutable in G. Therefore, we may assume $TL = G$.

T core free and $TL = G$ imply that T is a complement to L in G. Hence T is a Carter subgroup of G. T permutable in H and T self-normalizing in G yields $T = H$. Therefore we have $G \cap H = T$ completing the proof. □

Proof of Theorem A part (a) Suppose every subgroup of G is S-permutable sensitive in G. Using an argument similar to that used in the proof of part (b) of Theorem A, we can argue that the hypothesis is inherited by subgroups.

Now let us argue that G is a subgroup closed $\mathcal{PST}$-group. Let H be any subgroup of G and suppose N is S-permutable in K with K S-permutable in H. Then $N = K \cap L$ for some L S-permutable in H. By Lemma 1, we have $K \cap L$ is S-permutable in H. Thus N is S-per H and we have H is a $\mathcal{PST}$-group. Since subgroup closed $\mathcal{PST}$-groups are solvable $\mathcal{PST}$-groups (see Corollary 5 of [10]), G is a solvable $\mathcal{PST}$-group.

The argument for the converse is similar to the argument used in the proof of part (b) of Theorem A. □

Proof of Theorem B part (a) Let G be a group. Assume that G is a $\mathcal{PST}$-group and let K be S-permutable in U with U a normal subgroup of G. It is clear that U is S-permutable sensitive in G because K, being subnormal in G, is necessarily S-permutable in G.

Conversely, assume that every normal subgroup of G is S-permutable sensitive in G. Let U and V be subgroups of G such that $U \trianglelefteq V \trianglelefteq G$. Then there exists an S-permutable subgroup X of G such that $X \cap V = U$. By Lemma 1, U is S-permutable in G. Therefore, by Theorem A of [11], G is a $\mathcal{PST}$-group. □

Proof of Theorem B part (b) Let G be a group. Assume that G is a $\mathcal{PT}$-group and let K be permutable in U with U a subnormal subgroup of G. It is clear that U is permutable sensitive in G because K, being subnormal in G, is necessarily permutable in G.

Conversely, assume that every subnormal subgroup of G is permutable sensitive in G. As in the second paragraph of the proof of part (a), one can argue G is a $\mathcal{PST}$-group. If we assume that G is a p-group for some prime p, then every subgroup of G is permutable sensitive in G so that G is an Iwasawa group by part (b) of Theorem A.

Assume that G is not a p-group. Note that all the subnormal subgroups and all the factors groups of G satisfy the hypothesis of the theorem. By Lemma 3, G is an SC-group. Let D be the solvable residual of G and assume $D \neq 1$. Then G/D is a $\mathcal{PT}$-group. By Lemma 4, $D/Z(D) = U_1/Z(D) \times \cdots \times U_k/Z(D)$ where $U_i \trianglelefteq G$ and $U_i/Z(D)$ is a simple (nonabelian) group. Note that $U_i' \neq 1$ for all i. Therefore, if $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, k\}$ with $r \leq k$, then $G/U_{i_1}' U_{i_2}' \cdots U_{i_r}'$ is a $\mathcal{PT}$-group and hence satisfies N_p and P_p for all primes p. By Theorem 4, G is a $\mathcal{PT}$-group.

Now let us assume G is solvable. By Lemma 3, G is supersolvable and hence contains a normal Sylow p-subgroup P where p is the largest prime divisor of the order of G. Now both P and G/P are $\mathcal{PT}$-groups by induction. This means that P and the Sylow subgroups of G/P are Iwasawa groups. But G is a solvable $\mathcal{PST}$-group so it must be a $\mathcal{PT}$-group by Theorem 1. This completes the proof. □

The proof of part (c) of Theorem B is similar to the proof for part (a) of Theorem B and will be omitted.

Note how the proofs concerning $\mathcal{PT}$-groups in Theorems A and B are much more involved than the corresponding proofs for $\mathcal{PST}$-groups. This is directly related to the fact that permutable subgroups do not form a sublattice of the lattice of subnormal subgroups, whereas S-permutable subgroups do form a sublattice.

Proof of Theorem D (a). Assume that G is a $\mathscr{Y}_p$-group and let L, H, and K be p-subgroups of G with $L \leq H \leq K$. Then L is S-permutable in $N = N_G(K)$. Thus H is S-permutable sensitive in N.

Conversely, assume that G is a $\mathscr{Y}_p^*$-group and let H and K be p-subgroups of G with $H \leq K$. Then K is S-permutable sensitive in $N = N_G(K)$ and since H is S-permutable in K there exists an S-permutable subgroup L of N such that $H = L \cap K$. L and K are both S-permutable subgroups of N and hence, by Lemma 1, $H = L \cap K$ is S-permutable in N. It follows that G is a $\mathscr{Y}_p$-group.

(b). Throughout, let P be a Sylow p-subgroup of G and put $N = N_G(P)$.

Assume that G is an $\mathscr{X}_p^*$-group. First let us argue that P is an Iwasawa group. It is enough to show that P is a $\mathcal{PT}$-group. Let $H \leq P$. If C is a permutable subgroup of H, then $C = H \cap L$ with L permutable in N since G is an $\mathscr{X}_p^*$-group. Since $C = H \cap (P \cap L)$ and $P \cap L$ is permutable in P, we see that H is permutable sensitive in P. Applying part (b) of Theorem A, we see that P is a $\mathcal{PT}$-group and hence an Iwasawa group.

Let $X \leq P$. Since P is an Iwasawa group, we have X permutable in P. Thus, there exists a permutable subgroup Y of N such that $Y \cap P = X$. By Lemma 1, X is S-permutable in N. Now we can apply part (1) of Theorem 2 of [7] and deduce X is permutable in N. Thus G is an $\mathscr{X}_p$-group.

Conversely, assume that G is an $\mathscr{X}_p$-group. That G is an $\mathscr{X}_p^*$-group follows from the fact that P is an Iwasawa group.

(c). The proof of (c) is similar to that of (b). One simply replaces $\mathscr{X}_p$, $\mathscr{X}_p^*$, Iwasawa, and permutable, respectively, with $\mathscr{C}_p$, $\mathscr{C}_p^*$, Dedekind, and normal. Also, Theorem 3 is used in place of part (b) of Theorem A. □

Corollary 1 *$\mathscr{X}_p^*$ and $\mathscr{C}_p^*$ are subgroup-closed classes.*

Proof This follows from the fact that $\mathscr{X}_p$ (see Corollary 3 of [13] or Theorem 3 of [7]) and $\mathscr{C}_p$ (see the Corollary on page 936 of [23] or Theorem 3 and Lemma 1 of [7]) are subgroup-closed classes. □

Proofs of parts (a), (b), and (c) of Theorem C are, respectively, presented in [7], [13], and [23]. We will give a proof of part (a) of Theorem C different from that of [7]. Similar proofs for parts (b) and (c) of Theorem C are also possible, but will not be given here.

Proof of Theorem C part (a) Assume that G is a $\mathscr{Y}_p^*$-group for all primes p and that G is minimal with respect to not being a solvable $\mathcal{PST}$-group. Since $\mathscr{Y}_p^*$ is a subgroup-closed class for all p, we can assume G is a minimal-non-$\mathcal{PST}$-group. By Lemma 2 of [26], G has a normal Sylow p-subgroup P and a cyclic Sylow q-subgroup ($q \neq p$) such that G is a semidirect product of P by Q.

Let $X \leq P$. Since $N_G(P) = G$, P is S-permutable sensitive in G. So there exists an S-permutable subgroup W of G such that $W \cap P = X$. X is thus S-permutable in G by Lemma 1. Thus $XQ = QX$ and Q normalizes X since X is a subnormal Sylow p-subgroup of QX. Hence Q acts as a group of power automorphisms on P. By a result of Huppert [18], P must be abelian so that G is a $\mathcal{PST}$-group by Theorem 1, a contradiction.

Conversely, assume that G is a solvable $\mathcal{PST}$-group. Then the subgroups of G are also $\mathcal{PST}$-groups. Let K be a p-subgroup of G, p a prime, and let $N = N_G(K)$. Then $N_G(K)$ is a $\mathcal{PST}$-group so that if $H \leq K$, then H is S-permutable in N. Thus K is S-permutable sensitive in N and we see G is a $\mathscr{Y}_p^*$-group. □

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