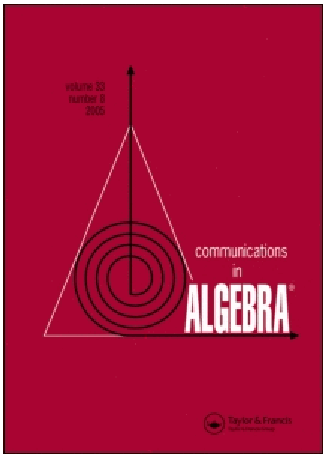

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Generalizations of Groups in which Normality Is Transitive

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GENERALIZATIONS OF GROUPS IN WHICH NORMALITY IS TRANSITIVE

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A group G is called a $\mathrm{Hall}_{\mathfrak{X}}$-group if G possesses a nilpotent normal subgroup N such that G/N' is an $\mathfrak{X}$-group. A group G is called an $\mathfrak{X}_o$-group if $G/\Phi(G)$ is an $\mathfrak{X}$-group. The aim of this article is to study finite solvable $\mathrm{Hall}_{\mathfrak{X}}$-groups and $\mathfrak{X}_o$-groups for the classes of groups $\mathcal{T}, \mathcal{PT}$, and $\mathcal{PST}$. Here $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$ denote, respectively, the classes of groups in which normality, permutability, and Sylow-permutability are transitive relations. Finite solvable $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups were globally characterized, respectively, in Gaschütz (1957), Zacher (1964), and Agrawal (1975). Here we arrive at similar characterizations for finite solvable $\mathrm{Hall}_{\mathfrak{X}}$-groups and $\mathfrak{X}_o$-groups where $\mathfrak{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. A key result aiding in the characterization of these groups is their possession of a nilpotent residual which is a nilpotent Hall subgroup of odd order. The main result arrived at is $\mathrm{Hall}_{\mathcal{PST}} = \mathcal{T}_o$ for finite solvable groups.

Key Words: Permutable; S-permutable; Solvable groups; Supersolvable groups; $\mathcal{T}$-groups.

Mathematics Subject Classification: 20Dxx; 20D10; 20D20; 20D35.

1. INTRODUCTION

Throughout this article, all groups considered will be finite. Notation and terminology not explained can be found in Robinson (1996) or Doerk and Hawkes (1992). This article will focus mainly on characterizing several classes of solvable groups. More specifically, we will look at several generalizations of the class of solvable groups in which normality is a transitive relation. First a few definitions and known results should be discussed.

A subgroup H is said to be *permutable (S-permutable)* in a group G if it satisfies the property $HK = KH$ for all subgroups (Sylow subgroups) K of G. H per G (H S-per G) denotes H is permutable (S-permutable) in G. Among the first to study permutable (S-permutable) subgroups was Ore (1939) (Kegel (1962)) who showed such subgroups are necessarily subnormal.

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$\mathcal{T}$-groups ($\mathcal{PT}$-groups, $\mathcal{PST}$-groups) are those groups G which possess a transitive normality (permutability, S-permutability) relation. That is, G is a $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) if for all subgroups H and K of G where $H \trianglelefteq K \trianglelefteq G$ (H per K per G, H S-per K S-per G) we have $H \trianglelefteq G$ (H per G, H S-per G). It is easily seen that $\mathcal{T}$-groups are those groups in which normality and subnormality coincide. It is a direct consequence of the subnormality of permutable (S-permutable) subgroups that $\mathcal{PT}$-groups ($\mathcal{PST}$-groups) are precisely those groups in which subnormality and permutability (S-permutability) coincide. Note that $\mathcal{T} \subseteq \mathcal{PT} \subseteq \mathcal{PST}$.

In the years 1957, 1964, and 1975, Gaschütz, Zacher, and Agrawal, respectively, proved the following definitive results on solvable $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups.

Theorem 1 (Gaschütz, 1957; Zacher, 1964; Agrawal, 1975). *A solvable $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) is supersolvable.*

Theorem 2 (Agrawal, 1975; Gaschütz, 1957; Zacher, 1964). *Let $L \trianglelefteq G$ with G a group satisfying the following:*

(i) *G/L is a $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group);*
(ii) *H subnormal in L implies $H \trianglelefteq G$;*
(iii) *L is a Hall subgroup of G.*

Then G is a $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group).

Theorem 3 (Agrawal, 1975; Gaschütz, 1957; Zacher, 1964). *Let G be a group with L the nilpotent residual of G. Then G is a solvable $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) if and only if the following conditions hold:*

(i) *L is a normal abelian Hall subgroup of G with odd order;*
(ii) *G/L is a Dedekind (Iwasawa[1], nilpotent) group;*
(iii) *G acts by conjugation as power automorphisms on L.*

In addition to these classic results, there have been a plethora of articles written characterizing $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups. See Alejandre et al. (2001), Asaad (2004), Ballester-Bolinches et al. (2003a,b), Ballester-Bolinches and Esteban-Romero (2001, 2002, 2003a,b), Beidleman et al. (1999), Bianchi et al. (2000), Bryce and Cossey (1989), and Robinson (1968) for additional information. Also, Robinson has given a nice survey of $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups in Robinson (2003).

Consider the following definitions.

Definition. $Hall_{\mathscr{X}}$ is the class containing all groups G possessing a normal nilpotent subgroup N such that G/N' is an $\mathscr{X}$-group.

Definition. $\mathscr{X}_o$ is the class containing all groups G where $G/\Phi(G)$ is an $\mathscr{X}$-group.

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[1] An Iwasawa group is one in which every subgroup is permutable.

$Hall_{\mathcal{T}}$-groups were introduced in Perez (2002) while $\mathcal{T}_o$-groups were introduced in van der Waall and Fransman (1996) and Fransman (1991). Several nice results concerning $Hall_{\mathcal{T}}$-groups and $\mathcal{T}_o$-groups were given in Perez (2002) and Fransman (1991) respectively, yet no characterizations of these classes were given. Also, $\mathcal{T}_o$-groups have further been studied in Asaad and Heliel (2001) and Ballester-Bolinches et al. (2005).

The main goal of this article is to study and characterize solvable $Hall_{\mathcal{X}}$-groups and $\mathcal{X}_o$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$ in the spirit of the aforementioned theorems of Gaschütz, Zacher, and Agrawal. It is clear from the definitions that each of these six classes can be considered a generalization of the class $\mathcal{T}$. In fact the following relationships are clear:

- $\mathcal{X} \subseteq Hall_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$;
- $\mathcal{X} \subseteq \mathcal{X}_o$ for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$;
- $Hall_{\mathcal{T}} \subseteq Hall_{\mathcal{PT}} \subseteq Hall_{\mathcal{PST}}$;
- $\mathcal{T}_o \subseteq \mathcal{PT}_o \subseteq \mathcal{PST}_o$.

2. BASIC LEMMAS

In this section, we give five useful lemmas, the first three of which will be used repeatedly without reference. The first lemma will allow for certain induction arguments.

Lemma 1. *If $\mathcal{X}$ is a homomorph, then $Hall_{\mathcal{X}}$ and $\mathcal{X}_o$ are homomorphs as well.*

Proof. Let G be a group with $H \trianglelefteq G$ and $\mathcal{X}$ a homomorph. First let us suppose G is a $Hall_{\mathcal{X}}$-group. By definition, there exits a normal nilpotent subgroup N of G such that G/N' is an $\mathcal{X}$-group. Thus $G/(NH)'H$ is an $\mathcal{X}$-group since $N' \leq (NH)'H$. Now since NH/H is a normal nilpotent subgroup of G/H and $(G/H)/(NH/H)' \simeq G/(NH)'H$, we have G/H is a $Hall_{\mathcal{X}}$-group. That $Hall_{\mathcal{X}}$ is a homomorph follows.

Now suppose G is an $\mathcal{X}_o$-group. Define A by $\Phi(G/H) = A/H$. By definition, $G/\Phi(G)$ is an $\mathcal{X}$-group. Now $(G/H)/\Phi(G/H) \simeq G/A$ which is an $\mathcal{X}$-group since $\Phi(G) \leq A$. Thus G/H is an $\mathcal{X}_o$-group and hence $\mathcal{X}_o$ is a homomorph. □

The next lemma says that one can essentially replace the normal nilpotent subgroup N with the Fitting subgroup in the definition of $Hall_{\mathcal{X}}$-groups provided that $\mathcal{X}$ is a homomorph.

Lemma 2. *Let G be a finite group with $F = \mathrm{Fit}(G)$ and let $\mathcal{X}$ be a homomorph. G/F' being an $\mathcal{X}$-group is both necessary and sufficient for G to be a $Hall_{\mathcal{X}}$-group.*

Proof. Since F is a normal nilpotent subgroup of G, the sufficiency is clear.

Now suppose G is a $Hall_{\mathcal{X}}$-group. Then there exists a normal nilpotent subgroup of G, say N, such that G/N' is an $\mathcal{X}$-group. Now $N' \leq F'$ and so G/F' is an $\mathcal{X}$-group. □

The following lemma will give sufficient conditions for a factor group to have a transitive normality, permutability, or S-permutability relation. It is a direct consequence of Theorem 2 and so its proof will be omitted.

Lemma 3. *Let L and N be normal subgroups of a group G. Then G/N is a $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group) if G satisfies the following:*

(i) *G/LN is a $\mathcal{T}$-group (PT-group, PST-group);*
(ii) *H/N subnormal in LN/N implies $H \trianglelefteq G$;*
(iii) *L is a Hall subgroup of G.*

The next result gives rise to the fact that the derived subgroup of a Hall complement to the nilpotent residual of a $\mathcal{PST}$-group is contained inside the group's Frattini subgroup. The result will be surprisingly useful, for if a $\mathcal{PST}$-group has a trivial Frattini subgroup, one can conclude the group has abelian Sylow subgroups. As a consequence of a result of Agrawal's (1975, Theorem 3.2), one can deduce that such a $\mathcal{PST}$-group is necessarily a $\mathcal{T}$-group.

Lemma 4. *Let G be a supersolvable group in which $L = \gamma_\infty G$, the nilpotent residual of G, is a Hall subgroup of G. A complement to L in G exists and if C is a complement to L in G, then $C' \le \Phi(G)$.*

Proof. Let G be a supersolvable group with $L = \gamma_\infty G$ a Hall subgroup of G. Then L must have a Hall complement in G, say C.

Let M be a maximal subgroup of G. Then M contains either L or some conjugate of C. Let us show that $C' \le M$ in either case.

If $L \le M$, then $M \trianglelefteq G$ for M/L is a maximal subgroup of the nilpotent group G/L. M has prime index in G since G is supersolvable. So G/M is cyclic and thus $G' \le M$ giving us $C' \le M$.

Now suppose $C^g \le M$ for some $g \in G$. G' is nilpotent since G is supersolvable and hence $C' \trianglelefteq G$ since C is a Hall subgroup of G (see Satz VI.9.10 in Huppert, 1967). Thus $C' \le M$.

We conclude $C' \le \Phi(G)$. □

Lemma 5. *For a group G, if $G/\Phi(G)$ is a solvable $\mathcal{PST}$-group, then $G/\Phi(G)$ is a $\mathcal{T}$-group.*

Proof. Let $G/\Phi(G)$ be a solvable $\mathcal{PST}$-group. Clearly, we can assume $\Phi(G) = 1$. So G is a $\mathcal{PST}$-group and we can write $G = L \rtimes C$ with $L = \gamma_\infty G$ an abelian Hall subgroup of G and C a complement to L in G. By Lemma 4, we have $C' \le \Phi(G)$ and hence C is abelian. Thus every Sylow subgroup of G is abelian and so, by Theorem 3.2 of Agrawal (1975), we have G is a $\mathcal{T}$-group completing the proof.

□

3. CHARACTERIZATION THEOREMS

In this section, we are interested in characterizing solvable $\mathcal{X}_o$-groups and $Hall_{\mathcal{X}}$-groups where $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. The following theorem, a direct implication of Lemma 5, shows the connection between the classes of solvable $\mathcal{X}_o$-groups and $Hall_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$.

Theorem 4. *In the realm of solvable groups, $Hall_{\mathcal{PST}} \subseteq \mathcal{T}_o = \mathcal{PT}_o = \mathcal{PST}_o$.*

Proof. Let G be a solvable $Hall_{\mathcal{PST}}$-group. Then G/F' is a $\mathcal{PST}$-group. Thus $G/\Phi(G)$ is a $\mathcal{PST}$-group since $F' \leq \Phi(G)$ and so, using Lemma 5, G is a $\mathcal{T}_o$-group.

It follows directly from Lemma 5 that solvable $\mathcal{PST}_o$-groups are $\mathcal{T}_o$-groups and hence $\mathcal{T}_o = \mathcal{PT}_o = \mathcal{PST}_o$ □

In view of the previous theorem, we will concentrate on $\mathcal{T}_o$ and make no mention of the classes $\mathcal{PT}_o$ and $\mathcal{PST}_o$ henceforth. The following result says solvable $\mathcal{T}_o$-groups are supersolvable and thus, by Theorem 4, so too are solvable $Hall_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$.

Theorem 5. *A solvable $\mathcal{T}_o$-group is supersolvable.*

Proof. If G is a solvable $\mathcal{T}_o$-group, then $G/\Phi(G)$ is a solvable $\mathcal{T}$-group. By Theorem 1, $G/\Phi(G)$ is supersolvable and hence G is supersolvable by Huppert's well-known result (Robinson, 1996, Theorem 9.4.5). □

Corollary 1. *Solvable $Hall_{\mathcal{X}}$-groups are supersolvable when $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$.*

Lemma 3 gives us a good starting point for characterizations of the groups under consideration. Any hope of having the converse of Lemma 3 hold for the case when N is the derived subgroup of the Fitting subgroup of G, or when N is the Frattini subgroup of G, lies in finding a normal Hall subgroup L so that (i) and (ii) of Lemma 3 hold true. The nilpotent residual seems a natural choice for L since one can easily see that (i) and (ii) of Lemma 3 hold using the aforementioned characterizations of Gaschütz, Zacher, and Agrawal. But is the nilpotent residual of such a group in question a Hall subgroup? The next theorem gives a positive answer to this question.

Theorem 6. *If G is a solvable $\mathcal{T}_o$-group, then $\gamma_\infty(G)$ is a nilpotent Hall subgroup of G of odd order.*

Proof. Let G be a solvable $\mathcal{T}_o$-group and let $L = \gamma_\infty(G)$. By Theorem 5, G is supersolvable. Thus L is certainly nilpotent of odd order.

If G is nilpotent, then $L = 1$ in which case L is considered a Hall subgroup of G. So let us assume G is not nilpotent.

Let $P \in \mathrm{Syl}_p(G)$ with p the largest prime divisor of $|G|$. By induction on $|G|$, LP/P is a Hall subgroup of G/P. Now if p does not divide $|L|$, we can conclude L is a Hall subgroup of G with a basic order argument. So let us suppose p divides $|L|$. Now assuming $O_{p'}(G) \neq 1$, we have $LO_{p'}(G)/O_{p'}(G)$ is a Hall subgroup of $G/O_{p'}(G)$. Now p divides $|LO_{p'}(G)/O_{p'}(G)|$ and so $PO_{p'}(G) \leq LO_{p'}(G)$. But $LO_{p'}(G) = O_p(L)O_{p'}(L)O_{p'}(G) = O_p(L)O_{p'}(G)$ and thus $P \leq O_p(L) \leq L$. So $LP/P = L/P$ is a Hall subgroup of G/P giving us L a Hall subgroup of G.

We can now suppose p divides $|L|$ and $O_{p'}(G) = 1$. So P is the Fitting subgroup of G and hence $L \leq P$. Now $G/\Phi(G)$ is a $\mathcal{T}$-group and so $L\Phi(G)/\Phi(G)$ is a Hall subgroup of $G/\Phi(G)$. $L\Phi(G)/\Phi(G)$ must be a p-group and so we can assume $L\Phi(G) = P$. If H is a Sylow p-complement to P in G, then $HL\Phi(G) = G$ giving us $HL = G$. This implies that L is a Sylow p-subgroup of G and thus certainly a Hall subgroup of G. □

Now we can easily apply Theorem 4 to arrive at the following corollary.

Corollary 2. *If G is a solvable* $Hall_{\mathcal{X}}$*-group for* $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$, *then* $\gamma_\infty(G)$ *is a nilpotent Hall subgroup of G of odd order.*

Using Theorems 3 and 6, Lemmas 3 and 5, and Corollary 2, the following characterization theorems for $\mathcal{T}_o$-groups and $Hall_{\mathcal{X}}$-groups for $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$ are immediate.

Theorem 7. *Let G be a solvable group. G is a* $\mathcal{T}_o$*-group if and only if G satisfies the following*:

(i) $\gamma_\infty(G)$ *is a nilpotent Hall subgroup of G of odd order*;
(ii) *G acts as power automorphisms on* $\gamma_\infty(G)\Phi(G)/\Phi(G)$.

Theorem 8. *Let G be a solvable group with* $F = \mathrm{Fit}(G)$ *and* $\mathcal{X} \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. *G is a* $Hall_{\mathcal{X}}$*-group if and only if G satisfies the following*:

(i) $\gamma_\infty(G)$ *is a nilpotent Hall subgroup of G of odd order*;
(ii) $G/\gamma_\infty(G)F'$ *is an* $\mathcal{X}$*-group*;
(iii) *G acts as power automorphisms on* $\gamma_\infty(G)F'/F'$.

The next theorem shows that in the realm of solvable groups, the classes $Hall_{\mathcal{PST}}$ and $\mathcal{T}_o$ are exactly the same providing us with an interesting characterization theorem.

Theorem 9. *G is a solvable* $Hall_{\mathcal{PST}}$*-group if and only if G is a solvable* $\mathcal{T}_o$*-group.*

Proof. By Theorem 4, only the sufficiency is in question. Let G be a solvable $\mathcal{T}_o$-group minimal with respect to not being a $Hall_{\mathcal{PST}}$-group. Let $L = \gamma_\infty(G)$ and $F = \mathrm{Fit}(G)$.

We can assume $F' = 1$ by choice of G. G is supersolvable by Theorem 5 and since $F' = 1$, we have L is abelian. Theorem 6 says L is a Hall subgroup of G. Throughout the proof, let C be a Hall complement to L in G. Then $C' \trianglelefteq G$ (see Satz VI.9.10 in Huppert, 1967). Suppose $C' \neq 1$ so that G/C' is a $Hall_{\mathcal{PST}}$-group by minimality. Now by Lemma 4, we have $C' \leq \Phi(G)$ and thus $\mathrm{Fit}(G/C') = F/C'$. Hence G/C' is a $\mathcal{PST}$-group since $F' = 1$. Using Theorem 3, G/C' acts by conjugation, and thus G too, as power automorphisms on $LC'/C' \overset{G}{\simeq} L$. So G acts by conjugation as power automorphisms on L. Thus G is a $\mathcal{PST}$-group by Theorem 3. So we can conclude $C' = 1$ and, as a consequence, G has abelian Sylow subgroups.

Let us show that all nontrivial factor groups of G are $\mathcal{T}$-groups. Let $A \trianglelefteq G$ with $A \neq 1$ so that G/A is a $Hall_{\mathcal{PST}}$-group. G has abelian Sylow subgroups and thus so does G/A. Being a nilpotent group, $\mathrm{Fit}(G/A)$ is abelian and so, since G/A is a $Hall_{\mathcal{PST}}$-group, we see G/A is a $\mathcal{PST}$-group. Hence G/A is a $\mathcal{T}$-group by Theorem 3.2 of Agrawal (1975). So we can conclude all nontrivial factor groups of G are $\mathcal{T}$-groups.

L is a Hall subgroup of G and G/L is a $\mathcal{T}$-group. So, if there does not exist a subgroup H of L with $H \ntrianglelefteq G$, then by Theorem 2, we have G is a $\mathcal{T}$-group contrary to the choice of G. Throughout the rest of the proof, let H denote a nonnormal subgroup of G contained in L. Note that H is subnormal in G, and furthermore, we can assume H is core free in G.

Suppose there exists two minimal normal subgroups of G, say M and N. G is supersolvable and so both M and N have prime order. Both HM and HN are normal in G since G/N and G/M are $\mathcal{T}$-groups. Now $H \neq HM$ and $H \neq HN$ so that H has prime index in both HM and HN giving H maximal in both HM and HN. It must then be the case that $HM \cap HN = HM = HN$. Thus $|M| = |N|$ and we can conclude that all minimal normal subgroups of G have the same prime order, say p. Furthermore, it must be the case that $F = L$ is the unique Sylow p-subgroup of G.

Let us now show G has at least two distinct minimal normal subgroups. Certainly G contains one minimal normal subgroup, say M. G/M is a $\mathcal{T}$-group, H is maximal in HM, and $HM \leq L$. Thus we have $HM \trianglelefteq G$, HM an abelian p-group, and $HM = H \times M$. C is a p'-group of automorphisms of the abelian p-group HM with M a C-invariant direct factor of HM. By Theorem 3.3.2 of Gorenstein (1980), there exists a subgroup K of G such that $HM = K \times M$ with K a C-invariant subgroup. Thus $K \trianglelefteq G$ and so either $K = 1$, in which case $HM = M$, or K contains another minimal normal subgroup of G. $HM = M$ implies H is not core free in G and so we can assume G does not have a unique minimal normal subgroup.

Now suppose G has three distinct minimal normal subgroups, say M, N, and K. It must be the case that each is a p-group contained in L. Also, we must have $HN = HM$, as was seen earlier. Thus $HN = HM = HMN$ and so $H \cap MN \neq 1$ by order considerations. G/K is a $\mathcal{T}$-group and so G/K, and thus G, acts by conjugation as power automorphisms on $\gamma_\infty(G/K) = L/K$. $MNK/K \leq L/K$, and so G acts as power automorphisms on the abelian p-group MNK/K and thus on MK/K and on NK/K. Let $g \in G$; then, using Theorem 13.4.3(ii) of Robinson (1996), g induces a power automorphism of the form $x \xrightarrow{g} x^z$ for all $x \in MNK/K$ with z a fixed integer. Now $MK/K \overset{G}{\simeq} M$ and $NK/K \overset{G}{\simeq} N$ and so g induces, by conjugation, power automorphisms in M and in N so that $m^g = m^z$ and $n^g = n^z$ for all $m \in M$ and for all $n \in N$. One can easily deduce that all subgroups of MN are normal in G. Thus $H \cap MN$ is normal in G which contradicts the fact that H is core free in G. So we can conclude that G has exactly two distinct minimal normal subgroups.

Throughout the rest of the proof, let M and N be the two distinct minimal normal subgroups of G. Let $HM = HN = Q$. $Q \trianglelefteq G$, Q is an abelian p-group, and $Q = H \times M = H \times N$. C is a p'-group of automorphisms of Q with M and N both C-invariant direct factors of Q. By Theorem 3.3.2 of Gorenstein (1980), there exist subgroups K_1 and K_2 of G such that $Q = K_1 \times M = K_2 \times N$ with K_1 and K_2 both C-invariant subgroups of G. So $K_1 \cap K_2 \trianglelefteq G$. Supposing $K_1 \cap K_2 \neq 1$, we must have $M \leq K_1 \cap K_2$ or $N \leq K_1 \cap K_2$. But $K_1 \cap M = K_2 \cap N = 1$. So we have $K_1 \cap K_2 = 1$. It is evident that $|K_1| = |K_2| = |H| = p^\beta$ for some integer β. Since $K_1K_2 \leq Q$ and $K_1 \cap K_2 = 1$, we have $p^{2\beta} \leq |Q| = p^{\beta+1}$ so that $\beta = 1$. Then H has order p and so all nonnormal subgroups of G contained in L have order p.

Let us show that L is elementary p-abelian. If C acts indecomposably on L, then, by Theorem 5.2.2 of Gorenstein (1980), we have L is homocyclic. Thus $L = L_1 \times L_2 \times \cdots \times L_l$ for some integer l with $|L_i| = |L_j|$ for all i and j. There must exist some $L_i \ntrianglelefteq G$, or else C acts decomposably on L. L_i being a nonnormal subgroup of G contained in L gives $|L_i| = p$, and thus L is elementary p-abelian when C acts indecomposably on L. If C does act decomposably on L, then $L = R \times S$ with R and

S both C-invariant and thus normal in G. C must act indecomposably on both R and S since G has exactly two minimal normal subgroups. Similarly, as was shown in the case when C acts indecomposably on L, we have R and S are elementary p-abelian and so L is elementary p-abelian.

Now suppose there exists a maximal subgroup of L, say T, which is not normal in G. Then T is of order p so that L is of order p^2. Thus $L = MN$. Now MC and NC have index p in G and are thus maximal in G. Hence $\Phi(G) \leq MC \cap NC$. Let $x = mc = nd \in MC \cap NC$ with $m \in M$, $n \in N$, and $c, d \in C$. Now $n^{-1}m = dc^{-1} \in L \cap C = 1$. So $n = m = 1$ since $M \cap N = 1$. Thus $x \in C$ so that $MC \cap NC \leq C$. Hence $\Phi(G) \leq C$ in which case $\Phi(G) = 1$ since $\Phi(G) \leq L$. But since G is a $\mathcal{T}_o$-group, we would then have G is a $\mathcal{T}$-group. This contradicts the minimality of G. So all maximal subgroups of L are normal in G.

Let $\{T_i : i \in I\}$ be the set of maximal subgroups of L with I some index set. T_iC has index p in G and so T_iC is maximal in G for all $i \in I$. Hence $\Phi(G) \leq \bigcap_{i\in I}(T_iC)$. Let $x \in \bigcap_{i\in I}(T_iC)$. Then for each $i \in I$, we can write $x = t_ic_i$ for some $t_i \in T_i$ and $c_i \in C$. $t_ic_i = t_jc_j$ implies $t_j^{-1}t_i = c_jc_i^{-1} \in L \cap C = 1$. So $t_i = t_j$ for any $i \neq j$. Thus $t_i \in \Phi(L)$ and hence $t_i = 1$ since L is elementary p-abelian. So $x \in C$ giving us $\bigcap_{i\in I}(T_iC) \leq C$. Therefore $\Phi(G) \leq C$ in which case $\Phi(G) = 1$. This gives a final contradiction to the minimality of G. Hence we can conclude that the class of solvable $Hall_{\mathcal{PST}}$-groups and the class of solvable $\mathcal{T}_o$-groups are precisely the same. □

The following lemma will allow for yet another characterization of solvable $\mathcal{T}_o$-groups.

Lemma 6. *Let G be a solvable $\mathcal{T}_o$-group. Then $L \cap F' = L'$ with $L = \gamma_\infty(G)$ and $F = \mathrm{Fit}(G)$.*

Proof. By Theorem 6, L is a nilpotent Hall subgroup of G. Thus L is a Hall subgroup of F. F is nilpotent and so $F = L \times H$ with H a Hall complement to L in F. So $F' = L'H'$ and $L \cap H' = 1$. Using Dedekind's modular law we have $L \cap F' = L \cap L'H' = L'(L \cap H') = L'$. □

Theorem 10. *Let G be a group with $L = \gamma_\infty(G)$. G is a solvable $\mathcal{T}_o$-group if and only if G/L' is a solvable $\mathcal{PST}$-group with L nilpotent.*

Proof. Let G be a solvable $\mathcal{T}_o$-group. Then L is a nilpotent Hall subgroup of G by Theorem 6. By Theorem 9, we have G is a solvable $Hall_{\mathcal{PST}}$-group. So G/F' is a $\mathcal{PST}$-group and hence G/F', and thus G, acts as power automorphisms on $\gamma_\infty(G/F') = LF'/F'$. Using Lemma 6, we have $LF'/F' \overset{G}{\simeq} L/L \cap F' = L/L'$. So G, and thus G/L', acts as power automorphisms on $L/L' = \gamma_\infty(G/L')$. L/L' is an abelian Hall subgroup of G/L' of odd order. Using Theorem 3, we see G/L' is a solvable $\mathcal{PST}$-group.

Now suppose G/L' is a finite solvable $\mathcal{PST}$-group with L nilpotent. G is certainly solvable and, by definition, G is a $Hall_{\mathcal{PST}}$-group. Theorem 9 says that G is a $\mathcal{T}_o$-group. □

Corollary 3. *G is a solvable $\mathcal{T}_o$-group with abelian nilpotent residual if and only if G is a solvable $\mathcal{PST}$-group.*

4. EXAMPLES

It is natural to ask if one needs to specify that L be nilpotent in Theorem 10. That is, if G/L' is a solvable $\mathcal{PST}$-group, then do we get L nilpotent for free? The following example shows the contrary.

Example 1. Let $G = S_4$ be the symmetric group on 4 elements. Then $\gamma_\infty(G) = A_4$, the alternating group on 4 elements, and $\gamma_\infty(G)' = K_4$, the Klein 4-group. $S_4/K_4 \simeq S_3$, the symmetric group on 3 elements. S_3 is clearly a $\mathcal{PST}$-group, yet $\gamma_\infty(G) = A_4$ is not nilpotent. Thus S_4 is not a $\mathcal{T}_o$-group. So one does need to specify that $\gamma_\infty(G)$ is nilpotent in Theorem 10.

Assuming we are working in the realm of finite solvable groups, the following diagram illustrates the relationship between the various classes discussed so far:

$$\begin{tikzcd}
\mathcal{T} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{PT} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{PST} \arrow[d, Rightarrow] & \\
Hall_{\mathcal{T}} \arrow[r, Rightarrow] & Hall_{\mathcal{PT}} \arrow[r, Rightarrow] & Hall_{\mathcal{PST}} \arrow[r, Leftrightarrow] & \mathcal{T}_{\mathrm{o}}
\end{tikzcd}$$

It is natural to ask whether or not the classes above are distinct. To finish the article, we will give two examples (which have been found through the use of GAP (2002)) that show the differences in the classes under consideration.

We will use the following groups in constructing our examples. A subscript on a group name indicates the order of that group:

$$C_i = \langle a \mid a^i = 1\rangle$$

$$D_8 = \langle r, s \mid r^4 = s^2 = 1, r^s = r^{-1}\rangle$$

$$G_{27} = \langle x, y, z \mid x^3 = y^3 = z^3 = 1, x^z = x^y = x, y^z = yx^{-1}\rangle = \big(\langle x\rangle \times \langle y\rangle\big) \rtimes \langle z\rangle$$

$$G_{64} = \left\langle v, w, y, z \,\middle|\, \begin{array}{l} y^8 = z^2 = w^2 = v^2 = 1, \\ w^v = w, z^v = z, y^v = yz, \\ y^w = y^5, z^w = z, y^z = y \end{array} \right\rangle = \big(\big(\langle y\rangle \times \langle z\rangle\big) \rtimes \langle v\rangle\big) \rtimes \langle w\rangle.$$

Example 2. The following example will show that solvable $Hall_{\mathcal{PST}}$-groups need not be $\mathcal{PST}$-groups nor $Hall_{\mathcal{PT}}$-groups.

Let

$$G_{216} = \left\langle r, s, x, y, z \,\middle|\, \begin{array}{l} r^4 = s^2 = x^3 = y^3 = z^3 = 1, r^s = r^{-1}, \\ x^z = x^y = x^s = x^r = x, y^z = yx^{-1}, \\ y^s = y^{-1}, z^s = z^{-1}, y^r = y, z^r = z \end{array} \right\rangle.$$

It is apparent from the generators and relations that there is a normal copy of $G_{27} = \langle x, y, z\rangle$ in G_{216} and a nonnormal copy of $D_8 = \langle r, s\rangle$ in G_{216}. So $G_{216} \simeq G_{27} \rtimes D_8$.

G_{27}/C_3 is abelian since $y^z C_3 = yx^{-1}C_3 = yC_3$ and so $G_{27}' = C_3$. G_{27}/C_3 is a Hall subgroup of G_{216}/C_3 since G_{27}/C_3 is a Sylow 3-subgroup. $(G_{216}/C_3)/(G_{27}/C_3) \simeq G_{216}/G_{27} \simeq D_8$ is nilpotent and thus a $\mathcal{PST}$-group. That G_{216}/C_3 acts by conjugation as power automorphisms on G_{27}/C_3 is clear from the relations and so any subgroup of G_{27}/C_3 is normal in G_{216}/C_3. So by Theorem 2, G_{216}/C_3 is a $\mathcal{PST}$-group. Thus G_{216} is a *Hall*$_{\mathcal{PST}}$-group. Now G_{216} is not a $\mathcal{PST}$-group or else, using Theorem 3, $\gamma_\infty(G_{216}) = G_{27}$ would be abelian which is not the case since $y^z = yx^{-1}$.

Let $C_4 = \langle r \rangle$ and let $F = \mathrm{Fit}G_{216}$. Then $F = G_{27} \times C_4$ and $F' = G'_{27} \times C'_4 = C_3$. If G_{216} is a *Hall*$_{\mathcal{PT}}$-group, then G_{216}/C_3 would be a $\mathcal{PT}$-group by Lemma 2. But then, by Theorem 3, we would have $(G_{216}/C_3)/(G_{27}/C_3) \simeq D_8$ an Iwasawa group which is not the case. So G_{216} is not a *Hall*$_{\mathcal{PT}}$-group.

Example 3. The following example will show that solvable *Hall*$_{\mathcal{PT}}$-groups need not be $\mathcal{PT}$-groups nor *Hall*$_{\mathcal{T}}$-groups.

Let

$$G_{192} = \left\langle x, y, z, w, v \;\middle|\; \begin{array}{l} x^3 = y^8 = z^2 = w^2 = v^2 = 1, x^w = x^{-1}, \\ x^y = x^z = x^v = x, w^v = w, z^v = z, \\ y^v = yz, y^w = y^5, z^w = z, y^z = y \end{array} \right\rangle.$$

It is apparent from the generators and relations that G_{192} contains a normal copy of $C_3 = \langle x \rangle$ and a nonnormal copy of $G_{64} = \langle y, z, w, v \rangle$. So $G_{192} \simeq C_3 \rtimes G_{64}$.

Let us first show that G_{192} is a *Hall*$_{\mathcal{PT}}$-group. The relations give rise to $F = \mathrm{Fit}(G_{192}) = C_3 \times \langle y, z, v \rangle$ and $F' = \langle z \rangle$. To show G_{192} is a *Hall*$_{\mathcal{PT}}$-group, we need to show G_{192}/F' is a $\mathcal{PT}$-group. Let $H = G_{192}/F' \simeq \langle x, y, w, v \rangle$. H acts as power automorphisms on $\langle x \rangle$ and $\langle x \rangle$ is an abelian Hall subgroup of H. Using Theorem 3, all we need to show is that $H/\langle x \rangle \simeq \langle y, w, v \rangle$ is an Iwasawa group to establish that G_{192} is a *Hall*$_{\mathcal{PT}}$-group. Through tedious calculation or by applying a theorem of Iwasawa's (see Iwasawa, 1941 or Theorem 2.3.1 of Schmidt, 1994) we see that $\langle y, w, v \rangle$ is indeed an Iwasawa group.

To see that G_{192} is not a *Hall*$_{\mathcal{T}}$-group it is enough to find a subnormal subgroup of G_{192}/F' which is not normal. $\langle yv \rangle F'/F'$ is subnormal in G_{192}/F' since $\langle yv \rangle \leq F$. However, wF' does not normalize $\langle yv \rangle F'/F'$ as one can check. The Sylow 2-subgroup of G_{192} is not an Iwasawa group and so G_{192} is not a $\mathcal{PT}$-group.

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