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Minimality and locally defined classes of groups

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Abstract The class of groups G for which every subgroup of $P \in \mathrm{Syl}_p(G)$ is normal (permutable) in $N_G(P)$ is called $\mathscr{C}_p$ ($\mathscr{X}_p$). A finite solvable group possesses a transitive normality (permutability) relation if and only if it is a $\mathscr{C}_p$-group ($\mathscr{X}_p$-group) for all primes p. The classes $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$ denote, respectively, the classes of groups in which normality, permutability, and S-permutability are transitive relations. Our main result shows that the minimal non-$\mathscr{C}_p$-groups and the minimal non-$\mathscr{X}_p$-groups, respectively, are just the minimal non-$\mathcal{T}$-groups and the minimal non-$\mathcal{PT}$-groups. In addition, we arrive a new characterization of the solvable $\mathcal{PT}$-groups and the solvable $\mathcal{PST}$-groups.

Keywords Permutably embedded · Permutable · S-permutable · $\mathcal{T}$-group · Subnormal

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1 Introduction

All groups considered will be finite.

For a group G, we say a subgroup H is permutable in G if $HK = KH$ for all subgroups K of G while H is said to be S-permutable in G if $HS = SH$ for all Sylow subgroups S of G. The classes $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$, will denote, respectively, the classes of groups in which normality, permutability, and Sylow-permutability are transitive relations. These classes of groups have been studied in a number of papers in recent years. In particular, we refer the reader to [1,3,4,7,9–11] for more information.

There have been many papers locally characterizing solvable $\mathcal{T}$-groups, $\mathcal{PT}$-groups, and $\mathcal{PST}$-groups. Of particular importance to us are the results found in [3,7,9]. Consider the following definitions.

Definition 1

1. A group G is a $\mathscr{C}_p$-group if each p-subgroup H of G contained in a Sylow p-subgroup P of G is normal in $N_G(P)$.
2. A group G is an $\mathscr{X}_p$-group if each p-subgroup H of G contained in a Sylow p-subgroup P of G is permutable in $N_G(P)$.
3. A group G is a $\mathscr{Y}_p$-group if for all p-subgroups H and K of G where $H \leq K \leq P$ with P a Sylow p-subgroup of G, one has H S-permutable in $N_G(K)$.

Note that $\mathscr{C}_p \subseteq \mathscr{X}_p \subseteq \mathscr{Y}_p$. The classes $\mathscr{C}_p$, $\mathscr{X}_p$, and $\mathscr{Y}_p$ are both subgroup-closed and quotient-closed classes (see [3,7,9]). It is known that a group G is a $\mathscr{C}_p$-group ($\mathscr{X}_p$-group, $\mathscr{Y}_p$-group) for all primes p if and only if G is a solvable $\mathcal{T}$-group ($\mathcal{PT}$-group, $\mathcal{PST}$-group). We refer the reader to [3,7,9] for the details.

Let $\mathcal{X}$ be a class of groups. By a minimal non-$\mathcal{X}$-group, we mean a group G all of whose subgroups belong to $\mathcal{X}$, yet $G \notin \mathcal{X}$. Ballester-Bolinches, Esteban-Romero, and Robinson proved that a group G is a minimal non-$\mathscr{Y}_p$-group for some prime p if and only if G is a minimal non-$\mathcal{PST}$-group (see Theorem 2 in [5]).

We establish the following corresponding results for the classes $\mathscr{C}_p$ and $\mathscr{X}_p$.

Theorem A

1. *A group G is a minimal non-$\mathscr{C}_p$-group for some prime p if and only if G is a minimal non-$\mathcal{T}$-group.*
2. *A group G is a minimal non-$\mathscr{X}_p$-group for some prime p if and only if G is a minimal non-$\mathcal{PT}$-group.*

The generalized Fitting subgroup of a group G is the set of all elements of G inducing inner automorphisms on all the chief factors of G. Asaad improved upon many of the known local characterizations of $\mathcal{T}$-groups by showing that certain properties only need hold for the primes dividing the order of the generalized Fitting subgroup (see Theorem 3.7 in [1]). In particular, Asaad proved that having the p-subgroups of G normally embedded for the primes p dividing the order of the generalized Fitting subgroup of G is equivalent to normality being transitive in G. We will use $F^*(G)$ to denote the generalized Fitting subgroup throughout and the reader is referred to X.13 of [8] for detailed information on $F^*(G)$.

Quite often when one arrives at a theorem characterizing $\mathcal{T}$-groups there are corresponding theorems characterizing both the $\mathcal{PT}$-groups and $\mathcal{PST}$-groups. We determine how Asaad's theorem (Theorem 3.7 in [1]) generalizes to the classes $\mathcal{PT}$ and $\mathcal{PST}$. Consider the following definitions.

Definition 2 Let U be a subgroup of a group G.

1. The subgroup U is said to be *permutably embedded* in G if each Sylow subgroup of U is a Sylow subgroup of some permutable subgroup of G.
2. The subgroup U is said to be *S-permutably embedded* in G if each Sylow subgroup of U is a Sylow subgroup of some Sylow-permutable subgroup of G.

We arrive at the following results.

Theorem B *Let G be a group.*

1. *G is a solvable $\mathcal{PT}$-group if and only if every p-subgroup of G is permutably embedded in G for all primes p dividing the order of $F^*(G)$.*
2. *G is a solvable $\mathcal{PST}$-group if and only if every p-subgroup of G is S-permutably embedded in G for all primes p dividing the order of $F^*(G)$.*

2 The Proofs

Proof (Proof of Theorem A) Let us prove part (2). A similar proof can be argued for part (1).

If G is a minimal non-$\mathcal{PT}$-group, then, by Theorem A in [7], G is not an $\mathscr{X}_p$-group for some prime p. However, every proper subgroup of G is a $\mathcal{PT}$-group and hence an $\mathscr{X}_p$-group by Theorem A in [7]. Thus G is a minimal non-$\mathscr{X}_p$-group.

Now suppose G is a minimal non-$\mathscr{X}_p$-group for some prime p. Then there exists a p-subgroup H contained in a Sylow p-subgroup P of G for which H is not permutable in $N_G(P)$. If $N = N_G(P) \neq G$, then N is an $\mathscr{X}_p$-group and so H is permutable in $N_N(P) = N$. This is contrary to the choice of H.

Let us suppose P is normal in G. If $P = G$, then G is clearly a minimal non-$\mathcal{PT}$-group. So we can assume $P \neq G$. Hence H is permutable in P. For the remainder of the proof, let q denote a prime different from p. Suppose that every element $z \in G$ with q-power order satisfies the property $P\langle z\rangle \neq G$. Then $P\langle z\rangle \in \mathscr{X}_p$ for all elements $z \in G$ with q-power order. Hence H is permutable in $N_{P\langle z\rangle}(P) = P\langle z\rangle$. Thus $H\langle z\rangle = \langle z\rangle H$ for all q-power elements z of G. It is easily deduced that this forces H to be permutable in G. So there must exist an element z of G of q-power order for which $P\langle z\rangle = G$.

Let us now show that $G \in \mathscr{X}_q$. Let $K \leq \langle z\rangle$ and let $R \in \mathrm{Syl}_p\big(N_G(\langle z\rangle)\big)$. Then $N_G(\langle z\rangle) = R\langle z\rangle$ and since R is subnormal in G, we have $R \trianglelefteq R\langle z\rangle$ which must give rise to z centralizing R. Thus K centralizes R so that $K \trianglelefteq R\langle z\rangle = N_G(\langle z\rangle)$. We can conclude $G \in \mathscr{C}_q$ and hence $G \in \mathscr{X}_q$.

Since $\mathscr{X}_r$ is a subgroup-closed class for any prime r we have that all proper subgroups of G are both $\mathscr{X}_p$-groups and $\mathscr{X}_q$-groups. Hence all proper subgroups of G are $\mathcal{PT}$-groups by Theorem A of [7]. Thus G is a minimal non-$\mathcal{PT}$-group.

Before we prove Theorem B, we need the following lemma.

Lemma 1 *If G satisfies the property that each of its p-subgroups is permutably embedded (S-permutably embedded) in G, then G is an $\mathscr{X}_p$-group ($\mathscr{Y}_p$-group). The converse holds provided G is p-solvable.*

Proof If G is p-solvable, then the converse follows from Theorem 3.3 in [11] (Theorem 3.1 in [11]).

Suppose that every p-subgroup of G is permutably embedded (S-permutably embedded) in G. By using induction on $|G|$ and Lemma 1 in [2] (Lemma 1 in [6]), we can assume that G is a minimal non-$\mathscr{X}_p$-group (non-$\mathscr{Y}_p$-group). Hence, by Theorem A (Theorem 2 in [5]), we have that G is a minimal non-$\mathcal{PT}$-group (non-$\mathcal{PST}$-group).

From the definition of permutably embedded (S-permutably embedded) it is clear that G is an $\mathscr{X}_p$-group ($\mathscr{Y}_p$-group) if G is a p-group. Now after applying Lemma 2 and Theorem 2 in [10] (Lemma 2 in [10]), we see $G = P \rtimes Q$ or $G = Q \rtimes P$ with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and the non-normal Sylow subgroups of G cyclic. If $G = Q \rtimes P$, then we see that $G/Q \simeq P \lneq G$ is an $\mathscr{X}_p$-group ($\mathscr{Y}_p$-group). By Lemma 4.1 in [1] (Lemma 6 in [4]), we have $G \in \mathscr{X}_p$ ($G \in \mathscr{Y}_p$).

So we can assume $G = P \rtimes Q$ with Q cyclic. Hence $G/P \simeq Q \in \mathscr{X}_q$ ($G/P \simeq Q \in \mathscr{Y}_q$). By Lemma 4.1 in [1] (Lemma 6 in [4]), we have $G \in \mathscr{X}_q$ ($G \in \mathscr{Y}_q$). But we have already shown, in the first paragraph of this proof, that if $G \in \mathscr{X}_q$ ($G \in \mathscr{Y}_q$) with G q-solvable, then the q-subgroups of G are permutably embedded (S-permutably embedded) in G. Thus all primary subgroups of G are permutably embedded (S-permutably embedded) in G. By Theorem 1.3 in [11] (Theorem 1.2 in [11]), we have that G is a solvable $\mathcal{PT}$-group ($\mathcal{PST}$-group). Hence, by Theorem A in [7] (Theorem 4 in [3]), G is an $\mathscr{X}_p$-group ($\mathscr{Y}_p$-group).

Proof (Proof of Theorem B) Let G be a solvable $\mathcal{PT}$-group ($\mathcal{PST}$-group). Then by Theorem 1.3 in [11] (Theorem 1.2 in [11]), we have that the p-subgroups of G are permutably embedded (S-permutably embedded) in G for all primes p and, in particular, for those primes p dividing $|F^*(G)|$.

Now suppose that for all primes p dividing $|F^*(G)|$, every p-subgroup of G is permutably embedded (S-permutably embedded) in G. Then by Lemma 1 we have that G is an $\mathscr{X}_p$-group ($\mathscr{Y}_p$-group) for all primes p dividing $|F^*(G)|$. Hence by Theorem 4.2 in [1] (Theorem B in [4]) we have G is a finite solvable $\mathcal{PT}$-group ($\mathcal{PST}$-group).

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