

On hypercentral factor groups from certain classes

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1 Introduction and statement of results

All groups considered here are finite. A T -group is a group G in which normality is a transitive relation, that is, if $H \triangleleft K \triangleleft G$, then $H \triangleleft G$. T -groups were studied by Gaschütz [21] and he proved

Theorem 1 ([21]). *Let R be the nilpotent residual of the group G . Then G is a solvable T -group if and only if R is an abelian Hall subgroup of G on which G acts by conjugation as power automorphisms and G/R is a Dedekind group.*

Throughout the rest of this paper, the hypercenter, nilpotent residual, Fitting subgroup and Frattini subgroup of G will be denoted by $Z_*(G)$, $\gamma_*(G)$, $\text{Fit}(G)$ and $\Phi(G)$ respectively. In [18] the first two authors introduce a new class of groups, the so-called T_1 -groups. A group G is a T_1 -group if $G/Z_*(G)$ is a T -group. Some of the basic properties of T_1 -groups are established in [18]. The present work is a continuation of [18].

In the light of [18, Theorem A] and Theorem 1, we begin with

Theorem A. *Let G be a group and let $R = \gamma_*(G)$ and $S = Z_*(G)$. Then G is a solvable T_1 -group if and only if the following hold:*

- (i) RS/S is an abelian Hall subgroup of G/S ;
- (ii) G acts by conjugation on R/R' as a group of power automorphisms.

A subgroup H of a group G is said to be *permutable* in G if $HK = KH$ for all subgroups K of G , and H is said to be *Sylow permutable* or *S -permutable* if $HS = SH$ for all Sylow subgroups S of G . Kegel [22] showed that an S -permutable subgroup of G is subnormal. From this it follows that S -permutability is a transitive relation in G (i.e. H is S -permutable in G whenever it is S -permutable in some S -permutable subgroup of G), precisely when every subnormal subgroup of G is S -permutable. Groups with this property are called *PST-groups*. Likewise groups in which permutability is

transitive, the so-called *PT*-groups, are precisely the groups in which every subnormal subgroup is permutable.

This means that *T*-groups form a subclass of the class of *PT*-groups which is in turn a subclass of the class of *PST*-groups. These classes have been studied in detail in [4], [9], [10], [11], [20], and [26].

Agrawal [1] and Zacher [29] prove the following:

Theorem 2. *Let R be the nilpotent residual of a group G .*

- (i) (Agrawal) *G is a solvable *PST*-group if and only if G acts by conjugation as a group of power automorphisms on R and R is an abelian Hall subgroup of G .*
- (ii) (Zacher) *G is a solvable *PT*-group if and only if it is a solvable *PST*-group and all subgroups of G/R are permutable.*

A group G is called a T_0 -group if $G/\Phi(G)$ is a *T*-group. Further, G is called a T'_0 -group (resp. a T''_0 -group) if $G/\Phi(G)$ is a *PT*-group (resp. a *PST*-group). T_0 -groups were introduced in [28] and were studied in detail in [12], [18], [24], [28].

Let \mathcal{S} denote the class of solvable groups and $T_0(\mathcal{S})$ the class of solvable T_0 -groups. Further, let $T'_0(\mathcal{S})$ (resp. $T''_0(\mathcal{S})$) denote the class of all solvable T'_0 -groups (resp. T''_0 -groups). In [24] the third author proved the following theorem.

Theorem 3. (i) $T_0(\mathcal{S}) = T'_0(\mathcal{S}) = T''_0(\mathcal{S})$.

- (ii) *Let G be a group with $R = \gamma_*(G)$. Then G is a $T_0(\mathcal{S})$ -group if and only if R is a nilpotent Hall subgroup of G and G acts by conjugation as a group of power automorphisms on $R\Phi(G)/\Phi(G)$.*

Let $T_1(\mathcal{S})$ denote the class of solvable T_1 -groups and let $T'_1(\mathcal{S})$ (resp. $T''_1(\mathcal{S})$) denote the class of all solvable groups G such that $G/Z_*(G)$ is a *PT*-group (resp. *PST*-group). A result similar to Theorem 3(i) is

Theorem B. $T_1(\mathcal{S}) = T'_1(\mathcal{S}) = T''_1(\mathcal{S})$.

By [18, Lemma 1], all subgroups of a solvable T_1 -group are T_1 -groups. This leads to our next statement.

Theorem C. *Let G be a group.*

- (i) *If all subgroups of G are T_1 -groups then G is supersolvable.*
- (ii) *The following statements are equivalent:*
 - (ii)₁ *G is a solvable T_1 -group;*
 - (ii)₂ *all subgroups of G are T_1 -group;*
 - (ii)₃ *G is a supersolvable T_1 -group.*

A similar result for *T*-groups is the following:

Theorem 4 ([19], [28]). *Let G be a group. The following statements are equivalent:*

- (i) G is a solvable T -group;
- (ii) all subgroups of G are T -group;
- (iii) G is a supersolvable T -group.

The next three theorems provide some indication on how close $T(\mathcal{S})$ is to $T_1(\mathcal{S})$.

Theorem D. *Let G be a group.*

- (i) *If all Sylow subgroups of G are T -groups and all subgroups of G are T_1 -groups, then $G/Z(G)$ is a T -group and $|G' \cap Z(G)| \leq 2$.*
- (ii) *Let $G/Z(G)$ be a solvable T -group and let $|G' \cap Z(G)| \leq 2$. If P is a Sylow p -subgroup of G , then P is abelian if p is odd and of class at most 2 if $p = 2$.*

To simplify the statement of the next theorem we introduce notation for a certain family of groups. Let p, q be primes such that q divides $p - 1$. We denote by $H(p, q, n)$ the extension of an elementary abelian p -group $\langle a, b \rangle$ of order p^2 by a cyclic group $\langle c \rangle$ of order q^n such that $H(p, q, n)$ is non-abelian, c centralizes a and c^q centralizes b . This describes the group up to isomorphism.

Theorem E. *The following statements are equivalent for a T_1 -group G :*

- (i) *All proper subgroups of G are T -groups;*
- (ii) *G satisfies one of the following conditions:*
 - (ii)₁ G is a solvable T -group;
 - (ii)₂ G is a minimal non-abelian p -group or a quaternion group Q_{16} ;
 - (ii)₃ G is isomorphic to some $H(p, q, n)$.

Theorem F. *The following statements are equivalent for a solvable T_1 -group G :*

- (i) *all proper quotient groups of G are T -groups;*
- (ii) *G satisfies one of the following conditions:*
 - (ii)₁ G is a T -group;
 - (ii)₂ G is a p -group, $|G'| = p$ and $Z(G)$ is cyclic;
 - (ii)₃ G is isomorphic to a non-abelian subgroup of $\text{Hol}(C_p) \times C_p$ for some prime p ;
 - (ii)₄ G is an extension of a group of type (ii)₂ above, of exponent p , by a cyclic group of order 2 such that $|G/G'| = 2$.

A study of the local properties of T -groups was initiated by Robinson in the seminal paper [25]. Further studies of local properties of T -groups, PT -groups and PST -groups have been carried out by many authors; see [2], [5], [6], [8], [9], [13], [14], [15].

Let p be a prime. A group G satisfies C_p if and only if each subgroup of a Sylow p -subgroup P of G is normal in the normalizer $N_G(P)$. Since subgroups of solvable

T -groups are T -groups, it follows that solvable T -groups are C_p -groups for all primes p . Robinson [25] proved

Theorem 5. *A group G is a solvable T -group if and only if G is a C_p -group for all primes p .*

Let G be a group and let Z_p be the Sylow p -subgroup of the hypercenter $Z_*(G)$ of G . A group G satisfies \bar{C}_p if and only if G/Z_p is a C_p -group. We now present a local version of solvable T_1 -groups.

Theorem G. *A group G is a solvable T_1 -group if and only if G is a \bar{C}_p -group for all primes p .*

A product of two groups A and B is called mutually permutable if $AX = XA$ and $YB = BY$ for all subgroups $X \leq B$ and $Y \leq A$. Mutually permutable products have been studied in detail in [7], [16], [17]. The following statement shows a connection between p -supersolvability and mutually permutable products.

Theorem H. *If G is a mutually permutable product of the p -supersolvable group A and the p -solvable \bar{C}_p -group B , then G^2 is p -supersolvable.*

Theorem G and Theorem H together have the following consequence:

Corollary. *If G is a mutually permutable product of the supersolvable group A and the solvable T_1 -group B , then G^2 is supersolvable.*

2 Preliminary results

In this section we provide certain facts and results that will be used in proving Theorems A–H.

Lemma 1. *Let G be a group and let $R = \gamma_*(G)$, $S = Z_*(G)$. Assume that R is nilpotent and RS/S is a Hall subgroup of $\text{Fit}(G/S)$. Then $\text{Fit}(G/S) = RS/S = \text{Fit}(G)/S$.*

Proof. Denote $\text{Fit}(G)$ by W . Since $S = Z_*(G)$ we have $\text{Fit}(G/S) = W/S$. Note that $RS/S \leq W/S$. Let H/S be a Hall subgroup of W/S such that $W/S = RS/S \times H/S$. Since $G/RS \cong (G/S)/(RS/S)$ is nilpotent, we have $RH/RS \leq Z_*(G/RS)$. Now

$$RH/RS = RSH/RS \cong H/H \cap RS = H/S$$

and this isomorphism is an operator isomorphism, showing that $H/S \leq Z_*(G/S)$ and $H \leq Z_*(G)$ since $S \leq Z_*(G)$. This proves Lemma 1. \square

Some well-known results concerning C_p are given in the next lemma and their proofs can be found in [8], [23], [25].

Lemma 2. *Let p be a prime. Then*

- (i) C_p is a subgroup-closed class.
- (ii) *Let M be a normal p' -subgroup of the group G . If G/M is a C_p -group, then so is G .*
- (iii) *If G is a C_p -group and $N \triangleleft G$, then G/N is a C_p -group.*

Lemma 3. *Let p be a prime. If G is a p -solvable \bar{C}_p -group, then G is p -supersolvable.*

Proof. Let V be a counter-example of minimal order. Then V has only one minimal normal subgroup; we call it M . We see that M must be a non-cyclic p -subgroup, and since $Z_*(V) = 1$ we have that V is a C_p -group. Let S be some Sylow p -subgroup of V . Then every subgroup of S is normal in $N(S)$ and S is a Dedekind group. Let K be a Hall p' -subgroup of V . If $KM \neq V$, then KM is a C_p -group by Lemma 2(i) and all subgroups of M are normal in KM and SM , so that M is cyclic by minimality, a contradiction. If $KM = V$, then M is the Sylow p -subgroup of V and $V = N(M)$; since V is a C_p -group, all subgroups of M are normal. Again, by minimality, M is cyclic, a contradiction. So Lemma 3 is true. \square

3 Proofs of the main results

Proof of Theorem A. First assume that G is a solvable T_1 -group. Then G/S is a T -group and by Theorem 1 the subgroup $\gamma_*(G/S) = RS/S$ is a normal abelian Hall subgroup of G/S ; also $R' \leq S$, and $[R, G] = R$ yields $S \cap R = R'$. By [18, Theorem A], G acts by conjugation on R/R' as a group of power automorphisms and (ii) holds.

Conversely, assume that (i) and (ii) hold for G . Then G is solvable, $R' \leq S$ and $\gamma_*(G/S) = RS/S$ is a normal abelian Hall subgroup of G/S . Since RS/S is G -isomorphic to $R/(R \cap S)$ and $R' \leq R \cap S$, it follows that the elements of G act as power automorphisms on RS/S . Now S is the hypercenter of G and so G is supersolvable. Thus G' is nilpotent. By Lemma 1 we have $RS/S = \text{Fit}(G/S) = G'S/S$ and hence $(G/S)/\gamma_*(G/S)$ is an abelian group. By Theorem 1, G/S is a T -group and so G is a T_1 -group. \square

Proof of Theorem B. Let G belong to $T_1''(S)$. By Theorem 2, G acts by conjugation on the normal abelian Hall subgroup $\gamma_*(G/S) = RS/S$ of G/S (here $R = \gamma_*(G)$ and $S = Z_*(G)$) as a group of power automorphisms. Hence G/S is supersolvable and so is G which means that G' is nilpotent. By Theorem 1, it remains to show that $(G/S)/\gamma_*(G/S)$ is abelian. By Lemma 1 we have $\text{Fit}(G/S) = RS/S$ and hence $G'S/S = RS/S = \gamma_*(G/S)$. Therefore $(G/S)/\gamma_*(G/S)$ is abelian. \square

Proof of Theorem C. By [18, Lemma 1], all solvable subgroups of G are supersolvable. A proposed minimal counter-example to (i) in Theorem C is a group with all proper subgroups supersolvable, and so is solvable by a result of Huppert (see [27, Theorem 10.3.4]), and hence supersolvable, contrary to construction. On the other

hand, by [18, Lemma 1], subgroups of solvable T_1 -groups are T_1 -groups. So Theorem C is true. \square

Proof of Theorem D(i). Assume that all Sylow subgroups of G are T -groups and that all of its subgroups are T_1 -groups. Then by Theorem C the group G is a supersolvable T_1 -group and its Sylow p -subgroups of odd order are abelian, while its Sylow 2-subgroups are Dedekind groups. The nilpotent residual R of G is nilpotent of odd order since G is supersolvable. It follows that R is abelian. On the other hand, G/R is a Dedekind group and $(G/R)' = G'/R$ is of order at most 2. Also G' is nilpotent so that G' is abelian. The hypercenter S of G is a Dedekind group and the Hall 2'-subgroup V of S is abelian and contained in $Z(G)$ since $[VR, G] = [V, G] \times R = R$. Therefore $S/Z(G)$ is a 2-group and $RZ(G)/Z(G)$ is an abelian Hall subgroup of $G/Z(G)$. By [18, Theorem A] and Theorem 1, $G/Z(G)$ is a T -group.

Let D be a Sylow 2-subgroup of G . Then $D' \leq D \cap G' \leq Z(G)$ and $DZ(G)/Z(G)$ is an abelian 2-group. Thus all Sylow subgroups of $G/Z(G)$ are abelian so that $|G' \cap Z(G)| \leq 2$ by a result of Taunt ([27, (10.1.7)]). \square

Proof of Theorem D(ii). Assume that $G/Z(G)$ is a solvable T -group and that $|G' \cap Z(G)| \leq 2$. By [18, Lemma 1], G is a supersolvable T_1 -group. Let p be a prime divisor of $|G|$ and let P be a Sylow p -subgroup of G . Assume that $p > 2$. Then $PZ(G)/Z(G)$ is abelian and $P' \leq G' \cap Z(G)$. But $|G' \cap Z(G)| \leq 2$ by hypothesis so that $P' = 1$ and so P is abelian.

We now consider $p = 2$. Then $|P'Z(G)/Z(G)| \leq 2$; also note that $|P' \cap Z(G)| \leq 2$. We want to show that $PZ(G)/Z(G)$ is abelian. Since $G/Z(G)$ is a solvable T -group, it is 2-nilpotent and $PZ(G)/Z(G)$ is a Dedekind group. If $PZ(G)/Z(G)$ is non-abelian, it is a direct product of a quaternion group $A \cong Q_8$ and an elementary abelian group B , and $P^2Z(G)/Z(G) = \langle u, Z(G) \rangle / Z(G)$ has order 2. Now $P^2Z(G) = P'Z(G)$ is a normal subgroup since $G/Z(G)$ is a T -group, and $P' \leq Z_*(G)$. If P' is cyclic, P must be nilpotent of class 3 and not of class 2. This yields that $P' \cap Z(G) = [P, P'] = [P, \langle u \rangle]$. However, since P has a basis of elements of (exact) order 4 and these elements all have square u , we have $[P, u] = 1$, a contradiction, and $u \in P' \cap Z(P)$. This shows that $PZ(G)/Z(G)$ must be abelian. Hence $P' \leq Z(G)$ and P is of class at most 2. \square

Proof of Theorem E. It is clear that (i) is a consequence of (ii).

Assume now that G satisfies (i). Then all proper subgroups of G are supersolvable, and so G is solvable. By [18, Lemma 1], G is supersolvable. Assume that G is not a T -group and that G is a p -group for $p > 2$. Then every subgroup is a T -group and thus abelian. This means that G is a minimal non-abelian group. Assume that G is a 2-group and not minimal non-abelian. Then G is a minimal non- T -group and by a result of Robinson [26] it is isomorphic to the quaternion group of order 16.

We have shown that a group G satisfying (i) but not (ii) is not a p -group. Note that G is a minimal non- T -group and $Z(G) \neq 1$. By Theorem 5 there is a prime p such that G is not a C_p -group. Let P be a Sylow p -subgroup and put $N = N_G(P)$. If $N \neq G$ then N is a T -group, which means that G is a C_p -group. Thus

$N = N_G(P) = G$. Let X be a Hall p' -subgroup of G . Then G is the semidirect product of P by X . By Theorem 1 there is an element $y \in X$ of order q^m which does not induce a power automorphism in P . Thus $\langle y, P \rangle$ is not a T -group, so that $G = \langle y, P \rangle$ and $X = \langle y \rangle$. Now G is supersolvable, so that p is the largest prime divisor of $|G|$ and hence P is abelian.

Let S be the hypercenter of G . Assume that S is a q -group. There is an element $t \in P$ such that $\langle t \rangle$ is not normal in G . Now G/S is a T -group, so that $\langle t \rangle S \triangleleft G$. But this means that $\langle t \rangle$ is a normal Sylow p -subgroup of $\langle t \rangle S$ so that $\langle t \rangle \triangleleft G$, a contradiction. Thus p divides $|S|$. Let $c \in Z(G)$ be an element of order p .

Now let W denote the subgroup of P generated by all of its elements of order p and assume that $W \neq P$. Then $W \triangleleft G$ and $\langle y, W \rangle$ is a T -group. By Theorem 1 the element y acts as a power automorphism on W . But $c^y = c$ and so $[W, y] = 1$. Now y induces a p' -automorphism of P which acts as the identity on $\Omega_1(P) = W$ and so acts as the identity also on P . This is a contradiction, so that $W = P$ and P is elementary abelian. There is an element $a \in P$ such that $\langle a \rangle$ is not normal in G . By Maschke's Theorem $P = \langle a_1 \rangle \times \cdots \times \langle a_t \rangle$ where each $\langle a_i \rangle$ is an irreducible $\langle y \rangle$ -module. Assume that $t > 2$. Then $\langle \langle a_1 \rangle \times \cdots \times \langle a_{t-1} \rangle, y \rangle$ and $\langle \langle a_2 \rangle \times \cdots \times \langle a_t \rangle, y \rangle$ are proper subgroups of G and hence T -groups. Now y induces power automorphisms in both $\langle a_1 \rangle \times \cdots \times \langle a_{t-1} \rangle$ and $\langle a_2 \rangle \times \cdots \times \langle a_t \rangle$, and since $a_2 \neq 1$ belongs to both of them, the power is the same in both cases; and all subgroups $\langle a \rangle \leq P$ are y -invariant, contrary to assumption. So $t = 2$ and $P = \langle a \rangle \times \langle b \rangle$. We may assume that $b \in Z(G)$. The element y centralizes $\langle b \rangle$; note also that $\langle P, y^q \rangle$ is a T -group, and so y^q acts as the identity mapping on $\langle a \rangle$. Therefore $G \cong H(p, q, m)$ and Theorem E is proved. \square

Proof of Theorem F. It is easy to verify that all groups in (ii) satisfy (i). Assume that G satisfies (i) and is not a T -group. The solvable T_1 -group G is supersolvable and all of its minimal normal subgroups are cyclic. Now $Z(G) \neq 1$ and G has a normal Sylow p -subgroup P , where p is the largest prime divisor of $|G|$.

Assume first that G has more than one minimal normal subgroup. Then by Theorem 1 and [18, Theorem A], the minimal normal subgroups have order p and $P = \text{Fit}(G)$. Assume further that p is odd. For every minimal normal subgroup N the group $\text{Fit}(G)/N$ is abelian, and so in our case $\text{Fit}(G)$ is abelian. Let R be the nilpotent residual of G and $N_1, N_2 \leq Z(G)$ be two minimal normal subgroups of G . Since $G/N_1, G/N_2$ are T -groups, we have $[P, G] \leq N_1 \cap N_2 = 1$ and G is abelian, which is excluded. So we may assume that $N_2 \not\leq Z(G)$ and $N_1 \leq Z(G)$. We deduce that G/N_2 operates by conjugation on P/N_2 as a group of power automorphisms and fixes $N_1 N_2 / N_2$ elementwise. So $[P, G] = N_2$. On the other hand, arguing via G/N_1 , we have $[G, P]N_1 = P$ and $P = N_1 \times N_2$. This shows that G is isomorphic to a non-abelian subgroup of $\text{Hol}(C_p) \times C_p$, and G satisfies (ii)₃. Now assume that $p = 2$. Then G is a 2-group and G/M is a Dedekind group for every proper normal subgroup M of G . If $|G'| = 2$, then G has a normal subgroup K of order 2 such that $K \cap G' = 1$ and G/K is Hamiltonian. Thus G must contain a subgroup of the form $Q_8 \times C_4$ where $G' = Q_8'$ and K is the subgroup of C_4 of order 2. There is a third minimal normal subgroup L of G contained in KG' , and G/L is mapped onto the central

product of Q_8 and C_4 , and this group is not a Dedekind group. So if G is a 2-group satisfying (i) which is not a T -group then it cannot have more than one minimal normal subgroup.

It remains to consider groups G with a unique minimal normal subgroup, N say. Note that $N \leq Z(G)$, $|N| = p$ and $\text{Fit}(G) = P$. Assume that $\text{Fit}(G) = G$. If p is odd, then G/N is abelian and $G' = N \leq Z(G)$ and $Z(G)$ is cyclic. This is case (ii)₂. If $p = 2$ and G/N is abelian we obtain (ii)₂ as in the case for odd primes. If $p = 2$, and G/N is non-abelian, then G/N is Hamiltonian. In this case $X/N = G'/N$ is of order 2 and G/N has a set of generators g_iN (each of order 4), such that $\langle (g_i)^2 \rangle N = X$. Now $[g_i, X] = 1$ for all i and $X \leq Z(G)$. Now G has only one minimal normal subgroup, and so X must be cyclic. Thus G is of nilpotency class 2 and $G/Z(G)$ is elementary abelian. But then G' is also elementary abelian, a contradiction. So $G' = N$ and G satisfies (ii)₂.

Assume now that $\text{Fit}(G) \neq G$ and that G has a unique minimal normal subgroup N . Then N must be a p -group for odd prime p and $F(G) = P$; also P/N is a T -group and $P' = N$ since N is the only minimal normal subgroup of G . So P satisfies (ii)₂. Since G/N is a T -group and $\text{Fit}(G)/N = \text{Fit}(G/N)$, which is a p -group, we have $G' = F(G)$ and the elements of G/N induce power automorphisms on P/N and the identity on N . The only non-identity power automorphism of P/N doing this is the inverting automorphism, and this is only possible if P is of exponent p . So $|G/P| = 2$ and G satisfies (ii)₄. Now all cases are treated, and we have proved Theorem F. \square

Proof of Theorem G. Assume that G is a \bar{C}_p -group for all primes p . We must show that $G/Z_*(G)$ is a solvable T -group. By Theorem 5 it is enough to show $G/Z_*(G)$ is a C_q -group for all primes q . Assume that q does not divide $|Z_*(G)|$. Then G is a C_q -group, so that $G/Z_*(G)$ is also a C_q -group. Now assume that q divides $|Z_*(G)|$ and let Z_q be the Sylow q -subgroup of $Z_*(G)$. Then G/Z_q is a C_q -group and by Lemma 2(iii) the group $(G/Z_q)/Z_*(G/Z_q)$ is a C_q -group. Since $Z_*(G/Z_q) = Z_*(G)/Z_q$ it follows that $G/Z_*(G)$ is a C_q -group and by Theorem 5, $G/Z_*(G)$ is a solvable T -group.

Conversely, assume that $G/Z_*(G)$ is a solvable T -group. We shall show that G is a \bar{C}_q -group for all primes q . Assume that q divides $|Z_*(G)|$. Note that G/Z_q is a solvable T_1 -group by [18, Lemma 1].

If q does not divide $|G/Z_q|$, then G/Z_q is a C_q -group and hence G is a \bar{C}_q -group. Suppose instead that q divides $|G/Z_q|$. Now q does not divide $|Z_*(G)/Z_q|$ and hence, by induction, G/Z_q is a \bar{C}_q -group. This means that G/Z_q is a C_q -group and G is a \bar{C}_q -group.

Next assume that q does not divide $|Z_*(G)|$. By Theorem 5 the group $G/Z_*(G)$ is a C_q -group. Since $Z_*(G)$ is a q' -group, G is a C_q -group by Lemma 2(ii). Thus G is a \bar{C}_q -group and the proof is complete. \square

Proof of Theorem H. By hypothesis, A is p -supersolvable, and so all chief factors of order divisible by p have order p . The group A induces on each of them a cyclic group of automorphisms of order dividing $p - 1$. So $A'A^{p-1}$ is p -nilpotent. On the other hand, the p -solvable \bar{C}_p -group B is p -supersolvable by Lemma 3, with the additional property that the normalizer $N(S) = N_G(S)$ of the Sylow p -subgroup S

induces power automorphisms on the Sylow p -subgroup and so $N(S)/C(S)$ is cyclic of order dividing $p - 1$.

First assume that $p = 2$. Then A is 2-nilpotent. Also B is 2-nilpotent. By [16, Theorem 4], G is 2-nilpotent and hence 2-supersolvable. Assume now that p is odd. By [16, Theorem 6], $G/(A \cap B)_G$ is p -supersolvable and so $(A \cap B)_G \neq 1$. Assume that G/N is p -supersolvable for all normal subgroups $N \neq 1$ of G . Then G has a unique minimal normal subgroup M , and we have $M \leq A \cap B$ and $\text{Fit}(G)$ is a p -group. Now $\text{Fit}(G)$ is supersolvably embedded in G if and only if $\text{Fit}(G)/\Phi(\text{Fit}(G))$ is supersolvably embedded in $G/\Phi(\text{Fit}(G))$. So $M \cap \Phi(\text{Fit}(G)) = 1$ and $\text{Fit}(G)$ is an elementary abelian p -group. By Maschke's Theorem $\text{Fit}(G)$ must be a direct product of normal subgroups and so M is the Sylow p -subgroup of G . Since M is the Sylow p -subgroup both of A and of B , the group $A/(C(M) \cap A)$ is abelian of exponent dividing $p - 1$ since A is p -supersolvable. Also $B/(C(M) \cap B)$ is cyclic of order dividing $p - 1$ since B is a p -supersolvable \bar{C}_p -group. So $B = \langle x, C(M) \cap B \rangle$ for some x . If x induces a power automorphism on M , then

$$G/C(M) = (AC(M)/C(M))(BC(M)/C(M)),$$

where the first factor $AC(M)/C(M) \cong A/(C(M) \cap A)$ is abelian of exponent dividing $p - 1$ and the second $BC(M)/C(M) \cong B/(C(M) \cap B)$ is of exponent dividing $p - 1$ and contained in the center of $G/C(M)$. So $G/C(M)$ is abelian of exponent dividing $p - 1$ and G is p -supersolvable, contradicting our assumption. This shows that x does not induce a power automorphism on M and we have $M = (C(x) \cap M) \times [x, M]$, with both direct factors non-trivial. Since B is a \bar{C}_p -group, x operates as a power automorphism on $[x, M]$. For every subgroup U with $C(M) \leq U \leq G$ we put $U^+ = U/C(M)$. Then $G^+ = A^+B^+$ is metabelian, and we choose a maximal abelian subgroup $D^+ \geq (G')^+$. We consider two cases: $x \in D$ and $x \notin D$.

If $x \in D$, then $D^+ \cap A^+ \leq Z(G^+)$, and by minimality of M we have that $Z(G^+)$ must be cyclic by Schur's Lemma. So $D = \langle y, x, C(M) \rangle$ with $y \in A$. Since x does not operate on M as a power automorphism, $x C(M)$ does not generate a normal subgroup of G^+ , and conjugates are contained in the abelian subgroup D^+ . On the other hand, y operates as a power automorphism on M since $y C(M) \in Z(G^+)$. A conjugate $s C(M)$ of $x C(M)$ must be an element $x^e y^f C(M)$; here y^f induces a power automorphism on M while x induces a power automorphism on $[x, M] \neq M$. The conjugate s of x similarly induces a power automorphism on $[s, M] \neq M$. If $x \not\equiv s$ modulo $C(M)$, the only possibility is $[x, M] = C(s) \cap M$ and $[s, M] = C(x) \cap M$, and so if y^{-g} induces the same power automorphism on $[x, M]$ as x , we have $s \equiv x^{-1} y^g$ modulo $C(M)$ as the only possibility. Since D^+ is self-centralizing in G^+ , $y C(M) \in Z(G^+)$ and $|G^+ : C_{G^+}(x C(M))| = 2$, we have $|G^+ : D^+| = 2$; also D^+ is abelian of exponent dividing $p - 1$. So G^2 is p -supersolvable in this case.

Assume now that $x \notin D$. Then x acts non-trivially by conjugation on the abelian group D^+ and M splits into $|\langle x, D \rangle / D|$ many D -invariant subspaces on which D acts as a group of power automorphisms, and differently on different subspaces. Now recall that $M = (C(x) \cap M) \times [x, M]$ and that x acts as a power automorphism

on $[x, M]$. Since x permutes the two D -invariant subspaces of M , we have that x inverts $[x, M]$ by conjugation and $x^2 \in C(M)$. Hence $D = A$ is p -supersolvable and $G^2 \leq A$. This completes the proof of Theorem H. \square

Examples. Let $p = 41$ and consider two extensions of the elementary abelian 41-group $\langle a, b \rangle$:

- (i) Choose $\langle c, d \rangle$ with $d^{-1}cd = c^9$ and $c^{20} = d^2 = 1$, and put

$$c^{-1}ac = a^2, \quad c^{-1}bc = b^{20}, \quad d^{-1}ad = b.$$

Then $A_1 = \langle a, b, c \rangle$ is 41-supersolvable and $B_1 = \langle a, b, d \rangle$ is a T_1 -group, and $A_1(B_1)^2 = A_1$ is 41-supersolvable.

- (ii) Choose $\langle e, f \rangle$ with $e^{40} = f^4 = 1$ and $e^{-1}fe = e^{10}f^{-1}$, and put

$$e^{-1}ae = b, \quad e^{-1}be = a^2, \quad f^{-1}af = a, \quad f^{-1}bf = b^9.$$

Then $A_2 = \langle a, b, e \rangle$ is 41-supersolvable and $B_2 = \langle a, b, f \rangle$ is a T_1 -group, and $(A_2)^2B$ is 41-supersolvable.

This shows that both situations considered in the proof of Theorem H really occur.

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