A Note on Finite $\mathcal{PST}$-Groups

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Definitions

What are $\mathcal{T}$-groups $\mathcal{PT}$-groups and $\mathcal{PST}$-groups?

- A $\mathcal{T}$-group is a group in which normality is a transitive relation.
- A $\mathcal{PT}$-group is a group in which permutability is a transitive relation.
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Three Locally Defined Classes

Definitions

- Let $\mathscr{C}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have $H \trianglelefteq N_G(P)$.
- Let $\mathscr{X}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have H per $N_G(P)$.
- Let $\mathscr{Y}_p$ be the class of groups G satisfying the property that, whenever $H \leq S \leq P \in \mathrm{Syl}_p(G)$, we have H S-per $N_G(S)$.

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Local Theorems for $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$

Theorem [Robinson 1968]

$G \in \mathscr{C}_p$ for all primes p if and only if G is a solvable $\mathcal{T}$-group.

Theorem [Beidleman, Brewster, Robinson 1999]

$G \in \mathscr{X}_p$ for all primes p if and only if G is a solvable $\mathcal{PT}$-group.

Theorem [Ballester-Bolinches, Esteban-Romero 2002]

$G \in \mathscr{Y}_p$ for all primes p if and only if G is a solvable $\mathcal{PST}$-group.

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More Local Theorems for $\mathcal{T}$ and $\mathcal{PT}$

Theorem [Asaad 2004]

G is a solvable $\mathcal{T}$-group if and only if $G \in \mathscr{C}_p$ for all primes p dividing the order of $F^*(G)$, the generalized Fitting subgroup of G.

Theorem [Asaad 2004]

G is a solvable $\mathcal{PT}$-group if and only if $G \in \mathscr{X}_p$ for all primes p dividing the order of $F^*(G)$, the generalized Fitting subgroup of G.

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Theorem ????

G is a solvable $\mathcal{PST}$-group if and only if $G \in \mathscr{Y}_p$ for all primes p dividing the order of $F^*(G)$, the generalized Fitting subgroup of G.

A Few Lemmas

Generalized Fitting Subgroup Results

- $F^*\big(F^*(G)\big) = F^*(G)$.
- If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.
- If $H \trianglelefteq G$, then $F^*(H) \leq F^*(G)$.

$\mathscr{Y}_p$ Results

- $G \in \mathscr{Y}_p$ if and only if G is either p-nilpotent or $G \in \mathscr{C}_p$ with the Sylow p-subgroups of G abelian.
- $G \in \mathscr{Y}_p$ with the Dedekind Sylow p-subgroups if and only if $G \in \mathscr{C}_p$.
- Let p be a prime with M a normal p'-subgroup of G. Then $G \in \mathscr{Y}_p$ if and only if $G/M \in \mathscr{Y}_p$.

The Proof

Claim

G is a solvable $\mathcal{PST}$-group if and only if $G \in \mathscr{Y}_p$ for all primes p dividing the order of $F^*(G)$, the generalized Fitting subgroup of G.

Outline of the Proof

- Suppose $G \in \mathscr{Y}_p \, \forall \, p \big| |F^*(G)|$.
- Argue $F^*(G) \lneq G$.
- $F^*\left(F^*(G)\right) = F^*(G) \Rightarrow F^*(G)$ is a solvable $\mathcal{PST}$-group.
- Deduce $F^*(G) = F(G)$.
- Assume $G \in \mathscr{C}_p \, \forall \, p \big| |F^*(G)|$.
- Argue G is a solvable $\mathcal{PST}$-group in this case.
- Deduce $\exists \, p \big| |F^*(G)|$ with G p-nilpotent and P a non-abelian Sylow p-subgroup.

Outline continued

- Argue $F^*(O_{p'}(G)) \leq F^*(G) \Rightarrow G \in \mathscr{Y}_p \, \forall \, p \big| |F^*(O_{p'}(G))|$.
- Deduce $O_{p'}(G) \in \mathscr{Y}_p \, \forall \, p \big| |F^*(O_{p'}(G))|$.
- Deduce $O_{p'}(G)$ is a solvable $\mathcal{PST}$-group.
- Deduce G is solvable.
- Let N be minimal normal in G contained in $O_p(G)$.
- Argue $[N, O_{p'}(G)] = 1$ and conclude $N = N \cap Z(P) \leq Z(G)$.
- Deduce $F^*(G/N) = F^*(G)/N = F(G)/N = F(G/N)$.
- Argue G/N is a solvable $\mathcal{PST}$-group.
- Conclude G is a solvable $\mathcal{PST}$-group.