The Intersection Map and Certain Classes of Finite Groups

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Definitions

What are $\mathcal{T}$-groups $\mathcal{PT}$-groups and $\mathcal{PST}$-groups?

- A $\mathcal{T}$-group is a group in which normality is a transitive relation.
- A $\mathcal{PT}$-group is a group in which permutability is a transitive relation.
- A $\mathcal{PST}$-group is a group in which Sylow-permutability is a transitive relation.

Or Equivalently...

- A $\mathcal{T}$-group is a group in which every subnormal subgroup is normal.
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Theorems of Gaschütz, Zacher, and Agrawal

Theorem [Gaschütz 1957]

G is a solvable $\mathcal{T}$-group if and only if the following hold for G:

- $\gamma_\infty(G)$ is an abelian Hall subgroup of G of odd order;
- G acts as power automorphisms on $\gamma_\infty(G)$;
- $G/\gamma_\infty(G)$ is a $\mathcal{T}$-group.

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$\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups Only Differ in their Sylows

Theorem [Agrawal 1975]

- G is a solvable $\mathcal{PST}$-group with Dedekind Sylow subgroups if and only if G is a solvable $\mathcal{T}$-group.
- G is a solvable $\mathcal{PST}$-group with Iwasawa (modular) Sylow subgroups if and only if G is a solvable $\mathcal{PT}$-group.

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Definitions

- Let $\mathscr{C}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have $H \trianglelefteq N_G(P)$.
- Let $\mathscr{X}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have H per $N_G(P)$.
- Let $\mathscr{Y}_p$ be the class of groups G satisfying the property that, whenever $H \leq S \leq P \in \mathrm{Syl}_p(G)$, we have H S-per $N_G(S)$.

Three Locally Defined Classes

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Theorem [Robinson 1968]

$G \in \mathscr{C}_p$ for all primes p if and only if G is a solvable $\mathcal{T}$-group.

Theorem [Beidleman, Brewster, Robinson 1999]

$G \in \mathscr{X}_p$ for all primes p if and only if G is a solvable $\mathcal{PT}$-group.

Theorem [Ballester-Bolinches, Esteban-Romero 2002]

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Local Theorems for $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$

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A Seemingly Forgotten Theorem

Definition

A subgroup H of a group G is said to be *normal sensitive* in G if the map $N \to H \cap N$ sends the lattice of normal subgroups of G onto the lattice of normal subgroups of H, that is, if

$$\{L | L \trianglelefteq H\} = \{H \cap N | N \trianglelefteq G\}.$$

Theorem [Bauman 1974]

G is a solvable $\mathcal{T}$-group if and only if every subgroup of G is normal sensitive in G.

Can this be Extended to $\mathcal{PT}$ and $\mathcal{PST}$?

Definitions

- A subgroup H of a group G is said to be *permutable sensitive* in G if
$$\{L | L \text{ per } H\} = \{H \cap N | N \text{ per } G\}.$$
- A subgroup H of a group G is said to be *S-permutable sensitive* in G if
$$\{L | L \text{ S-per } H\} = \{H \cap N | N \text{ S-per } G\}.$$

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Theorem ????

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Theorem [Beidleman and Ragland]

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Restricting S-permtable, permutable, and normal sensitivity to the subnormal subgroups of G

Theorem [Beidleman and Ragland]

Let G be a group. Then

- G is a $\mathcal{PST}$-group if and only if every normal subgroup of G is S-permutable sensitive in G.
- G is a $\mathcal{PT}$-group if and only if every subnormal subgroup of G is permutable sensitive in G.
- G is a $\mathcal{T}$-group if and only if every normal subgroup of G is normal sensitive in G.

Open Question

It is still an open question whether or not one can replace “subnormal” with “normal” in the $\mathcal{PT}$ case above.

A Local Approach to Sensitivity; Definitions

Definition

- G is a $\mathscr{Y}_p^*$-group if, for each p-subgroup K of G, each subgroup H of K is S-permutable sensitive in $N_G(K)$.
- G is an $\mathscr{X}_p^*$-group if each subgroup of a Sylow p-subgroup P is permutable sensitive in $N_G(P)$.
- G is a $\mathscr{C}_p^*$-group if each subgroup of a Sylow p-subgroup P is normal sensitive in $N_G(P)$.

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A Local Approach to Sensitivity; Theorems

Theorem [Beidleman and Ragland]

- G is a solvable $\mathcal{PST}$-group if and only if G is a $\mathscr{Y}_p^*$-group for all primes p.
- G is a solvable $\mathcal{PT}$-group if and only if G is a $\mathscr{X}_p^*$-group for all primes p.
- G is a solvable $\mathcal{T}$-group if and only if G is a $\mathscr{C}_p^*$-group for all primes p.

Theorem [Beidleman and Ragland]

- $\mathscr{Y}_p = \mathscr{Y}_p^*$
- $\mathscr{X}_p = \mathscr{X}_p^*$
- $\mathscr{C}_p = \mathscr{C}_p^*$

A Local Approach to Sensitivity; Theorems

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- $\mathscr{Y}_p = \mathscr{Y}_p^*$
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Theorem

G is a solvable $\mathcal{PT}$-group if and only if every subgroup of G is permutable sensitive in G.

A Proof

$\Rightarrow$

- G is a solvable $\mathcal{PT}$-group with nilpotent residual L.
- $G = L \rtimes C$ with C a Carter subgroup.
- Both L and C are Hall subgroups.
- Let T per $H \leq G$. Assume T is corefree.
- Thus T per $H \cap TL$.
- Assume $TL \neq G$.
- Induct. $\exists K$ per TL such that $H \cap TL \cap K = H \cap K = T$
- TL per G since G/L is a nilpotent $\mathcal{PT}$-group.
- K per TL per $G \Rightarrow K$ per G.
- Thus H is permutable sensitive in G.
- So assume $TL = G$.
- So T is a complement to L in G.
- Hence T is a Carter subgroup of G.
- T sn H and T is self normalizing in G yield $T = H$.
- Therefore we have $G \cap H = T$.

$\Leftarrow$

- Suppose every subgroup of G is permutable sensitive in G.
- Also, pick G minimal with respect to not being a $\mathcal{PT}$-group.
- Deduce every proper subgroup of G is a $\mathcal{PT}$-group.
- Deduce every proper quotient of G is a $\mathcal{PT}$-group.
- Deduce G is solvable.
- Argue the case when G is a p-group.
- Apply a result of Robinson and write $G = P \rtimes Q$.
- Here P is Dedekind and Q is cyclic.
- $\exists A$ sn G with $A \not\text{per}\, G$.
- Assume A is corefree.
- If A is a q-group, then $A \trianglelefteq G$.
- $\exists A_p \in \mathrm{Syl}_p(A)$. Note $A_p \trianglelefteq P$.
- $\exists T$ per G with $P \cap T = A_p$.
- But T per $G \Rightarrow QT = TQ = A_pQ$ yielding $A_p \trianglelefteq G$.

A Few Questions?

Questions

1. What can be said about groups in which every subgroup is subnormal sensitive?
2. What can be said about groups G in which $\{L | L \trianglelefteq H\} = \{H \cap N | N \text{ per } G\}$ holds for all subgroups H of G?
3. What can be said about groups G in which $\{L | L \trianglelefteq H\} = \{H \cap N | N \text{ S-per } G\}$ holds for all subgroups H of G?