

# Strong Sylow Bases and Mutually Permutable Products

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Dedicated to Prof. Derek Robinson on the occasion of his 70<sup>th</sup> birthday



# Permutability Concepts

All groups are finite.

- A subgroup  $H$  of a group  $G$  is said to *permute* with the subgroup  $K$  if  $HK$  is a subgroup of  $G$ .
- $H$  is said to be *permutable (S-permutable)* in  $G$  if it permutes with all the subgroups (Sylow subgroups) of  $G$ .
- Subgroups  $H$  and  $K$  of  $G$  are said to be *mutually permutable* in  $G$  provided that  $H$  permutes with every subgroup of  $K$  and  $K$  permutes with every subgroup of  $H$ .
- $H$  and  $K$  are said to be *totally permutable* in  $G$  if every subgroup of  $H$  permutes with every subgroup of  $K$ .



- A set of pairwise permutable Sylow subgroups of a group  $G$ , one for each prime dividing the order of  $G$ , is called a *Sylow basis* of  $G$ .
- Let  $\{p_1, p_2, \dots, p_r\}$  be the set of prime divisors of  $|G|$  and let  $\{P_1, P_2, \dots, P_r\}$  be a Sylow basis of  $G$  with  $P_i \in \text{Syl}_{p_i}(G)$  for all  $i$ .
- If  $\pi$  is a subset of  $\{p_1, p_2, \dots, p_r\}$ , then  $H = \prod_{p_i \in \pi} P_i$  is a Hall  $\pi$ -subgroup of  $G$ , that is  $([G : H], |H|) = 1$ .
- A second Sylow basis  $\{Q_1, Q_2, \dots, Q_r\}$  is said to be conjugate to  $\{P_1, P_2, \dots, P_r\}$  if there is an element  $g \in G$  such that  $P_i^g = Q_i$ ,  $1 \leq i \leq r$ .

## Theorem (P. Hall)

- *A group  $G$  is solvable if and only if it has a Sylow basis.*
- *In a solvable group  $G$  any two Sylow bases are conjugate.*
- *In a solvable group  $G$  every  $\pi$ -subgroup,  $\pi$  a set of primes, is contained in a Hall  $\pi$ -subgroup. Moreover all the Hall  $\pi$ -subgroups of  $G$  are conjugate.*

- O. Kegel proved that an S-permutable subgroup is subnormal.
- We say that S-permutability is a transitive relation in a group  $G$  if when a subgroup  $H$  that is S-permutable in an S-permutable subgroup  $K$  of  $G$  is also S-permutable in  $G$ .
- A group  $G$  is called a  $\mathcal{PST}$ -group if S-permutability is a transitive relation in  $G$ . By Kegel's result  $\mathcal{PST}$ -groups are exactly those groups in which all subnormal subgroups are S-permutable.

# $\mathcal{T}$ , $\mathcal{PT}$ , and $\mathcal{PST}$ - groups

- Likewise, permutability is a transitive relation in  $G$  if every subnormal subgroup of  $G$  is permutable. (O. Ore)
- Groups in which permutability is a transitive relation are called  $\mathcal{PT}$ -groups.
- Examples of  $\mathcal{PT}$ -groups include the *Iwasawa* groups, the groups all of whose subgroups are permutable.
- A group  $G$  is called a  $\mathcal{T}$ -group if normality is a transitive relation in  $G$ . *Dedekind* groups, that is, groups all of whose subgroups are normal, are examples of  $\mathcal{T}$ -groups.



## Theorem

- (Agrawal) *A solvable group is a  $\mathcal{PST}$ -group if and only if it has a normal abelian Hall subgroup  $L$  such that  $G/L$  is nilpotent and  $G$  acts by conjugation on  $L$  as power automorphisms.*
- (Zacher) *A solvable group is a  $\mathcal{PT}$ -group if and only if it is a solvable  $\mathcal{PST}$ -group whose Sylow subgroups are Iwasawa.*
- (Gaschütz) *A solvable group is a  $\mathcal{T}$ -group if and only if it is a solvable  $\mathcal{PST}$ -group whose Sylow subgroups are Dedekind.*

The subgroup  $L$  in the previous theorems may be taken to be the nilpotent residual of  $G$ .

# $\mathcal{T}$ , $\mathcal{PT}$ , and $\mathcal{PST}$ - groups

- We note that the classes of finite solvable  $\mathcal{T}$ ,  $\mathcal{PT}$ , and  $\mathcal{PST}$ -groups are closed under forming quotients and subgroups.
- Also, the class of nilpotent groups is a subclass of the solvable  $\mathcal{PST}$ -groups.

Theorem (Asaad, Ballester-Bolinches, Beidleman, Esteban-Romero)

*Let  $G = G_1 G_2 \cdots G_k$  be the product of pairwise mutually permutable solvable  $\mathcal{PST}$ -subgroups  $G_1, G_2, \dots, G_k$ . Then  $G$  is supersolvable. In particular, if  $G_1, G_2, \dots, G_k$  are nilpotent, then  $G$  is supersolvable.*

Question?

What can one say about a group possessing a **set of Sylow subgroups** whose members are not only pairwise permutable, as in a Sylow basis, but are **pairwise mutually permutable**?

# Strong Sylow Bases

- Let  $\{P_1, \dots, P_k\}$  be a Sylow basis for the solvable group  $G$ . Then  $G = P_1 P_2 \cdots P_k$  is the product of the pairwise permutable subgroups  $P_1, P_2, \dots, P_k$ .
- If the subgroups  $P_1, \dots, P_k$  are pairwise mutually permutable, then  $\{P_1, \dots, P_k\}$  is called a *strong Sylow basis* for  $G$ .
- By the previous theorem, we see that a group with a strong Sylow basis is supersolvable.



## Theorem

*Let  $G$  be a group.*

- $G$  is a solvable  $\mathcal{PST}$ -group if and only if it has a strong Sylow basis.*
- If  $\{P_1, \dots, P_k\}$  is a strong Sylow basis for  $G$ , then the subgroups  $P_1, \dots, P_k$  are pairwise totally permutable.*
- If  $G$  has a strong Sylow basis, then every Sylow basis is strong.*
- If  $G$  has a strong Sylow basis, then so does every subgroup of  $G$ .*

## Theorem

- A group  $G$  is a solvable  $\mathcal{PT}$ -group if and only if it has a strong Sylow basis and its Sylow subgroups are Iwasawa.*
- A group  $G$  is a solvable  $\mathcal{T}$ -group if and only if it has a strong Sylow basis and its Sylow subgroups are Dedekind.*

# The Connection with System Normalizers

Theorem (Ballester-Bolinches, Cossey, Escriva-Soler)

*Let  $G$  be a solvable group.*

- *If a subgroup  $H$  of  $G$  permutes with each system normalizer of  $G$ , then  $H$  is subnormal in  $G$ .*
- *The group  $G$  is a  $\mathcal{PST}$ -group if and only if every subnormal subgroup of  $G$  permutes with all the system normalizers of  $G$ .*

Theorem (Ballester-Bolinches, Cossey, Esteban-Romero)

*If  $G = AB$  is the totally permutable product of the two nilpotent subgroups  $A$  and  $B$ , then  $G$  has an abelian nilpotent residual.*



# The Connection with System Normalizers

Since we have characterized the solvable  $\mathcal{PST}$ -groups in terms of their Sylow bases and mutual permutability, it only seems natural that there is a way to characterize them in terms of their system normalizers and permutability. This is the result of the next theorem.

## Theorem

*Let  $G$  be solvable with  $D$  a system normalizer of  $G$ .*

- $G$  is a  $\mathcal{PST}$ -group if and only if  $D$  is a Hall subgroup with Hall complement  $L$  for which every subgroup of  $L$  permutes with  $D$ .*
- $G$  is a  $\mathcal{PT}$ -group if and only if  $D$  is an Iwasawa Hall subgroup with Hall complement  $L$  for which every subgroup of  $L$  permutes with  $D$ .*
- $G$  is a  $\mathcal{T}$ -group if and only if  $D$  is a Dedekind Hall subgroup with Hall complement  $L$  for which every subgroup of  $L$  permutes with  $D$ .*

# The Connection with System Normalizers

The following corollary is immediate.

## Corollary

*Let  $G$  be solvable with  $D$  a system normalizer of  $G$ .*

- $G$  is a  $\mathcal{PST}$ -group if and only if  $D$  is a Hall subgroup with Hall complement  $L$  where  $G = LD$  is a mutually permutable product of  $L$  with  $D$ .*
- $G$  is a  $\mathcal{PT}$ -group if and only if  $D$  is an Iwasawa Hall subgroup with Hall complement  $L$  where  $G = LD$  is a mutually permutable product of  $L$  with  $D$ .*
- $G$  is a  $\mathcal{T}$ -group if and only if  $D$  is a Dedekind Hall subgroup with Hall complement  $L$  where  $G = LD$  is a mutually permutable product of  $L$  with  $D$ .*

# The Connection with System Normalizers

Having a strong Sylow basis forces a lot of permutability conditions on the Sylow subgroups. It is reasonable to ask if some of these permutability conditions can be relaxed and still yield a  $\mathcal{PST}$ -group.

## Definition

Let  $L$  denote the nilpotent residual of  $G$ . Let us call  $\Sigma = \{P_1, P_2, \dots, P_k\}$  a **strong Sylow system** of  $G$  if  $\Sigma$  consists of a set of Sylow subgroups of  $G$ ,  $P_i \in \text{Syl}_{p_i}(G)$ , where for each  $P_i \in \Sigma$  with  $p_i \in \pi(L)$  we have  $P_i P_j$  is a mutually permutable product for all  $P_j \in \Sigma$ .

## Theorem

*Let  $G$  be a group. Then  $G$  is a solvable  $\mathcal{PST}$ -group if and only if  $G$  possesses a strong Sylow system.*

Analogous results for  $\mathcal{T}$  and  $\mathcal{PT}$  results hold by putting the necessary conditions on the Sylow subgroups.

## Definition

Let  $p$  be a prime. We say a group  $G$ :

- has property  $\mathcal{C}_p$  if each subgroup of a Sylow  $p$ -subgroup  $P$  of  $G$  is normal in the normalizer  $N_G(P)$ ;
- has property  $\mathcal{X}_p$  if each subgroup of a Sylow  $p$ -subgroup  $P$  of  $G$  is permutable in the normalizer  $N_G(P)$ ;
- has property  $\mathcal{Y}_p$  if for all  $p$ -subgroups  $H$  and  $K$  of  $G$  such that  $H \leq K$ ,  $H$  is S-permutable in the normalizer  $N_G(K)$ .

**Note:** The classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$  and  $\mathcal{Y}_p$  are subgroup and quotient closed.



## Theorem

Let  $G$  be a group.

- (Robinson)  $G$  is a solvable  $T$ -group if and only if  $G$  is a  $\mathcal{C}_p$ -group for all primes  $p$ .
- (Beidleman, Brewster and Robinson)  $G$  is a solvable  $\mathcal{PT}$ -group if and only if  $G$  is a  $\mathcal{X}_p$ -group for all primes  $p$ .
- (Ballester-Bolinches and Esteban-Romero)  $G$  is a solvable  $\mathcal{PST}$ -group if and only if  $G$  is a  $\mathcal{Y}_p$ -group for all primes  $p$ .

## Theorem (Ballester-Bolinches and Esteban-Romero)

Let  $p$  be a prime and  $G$  a  $\mathcal{Y}_p$ -group.

- $G$  is an  $\mathcal{X}_p$ -group if the Sylow  $p$ -subgroups of  $G$  are Iwasawa.
- $G$  is a  $\mathcal{C}_p$ -group if the Sylow  $p$ -subgroups of  $G$  are Dedekind.

# Classes $\mathcal{C}_p$ , $\mathcal{X}_p$ and $\mathcal{Y}_p$

We next consider the relationship of the classes  $\mathcal{C}_p$ ,  $\mathcal{X}_p$  and  $\mathcal{Y}_p$  with strong Sylow bases.

## Definition

Let  $p$  be a prime.

- A group  $G$  has property  $\overline{\mathcal{Y}}_p$  if whenever  $K$  is a  $p$ -subgroup of  $G$ ,  $N_G(K)$  has strong Sylow basis.
- A group  $G$  has property  $\overline{\mathcal{X}}_p$  if it satisfies  $\overline{\mathcal{Y}}_p$  and has Iwasawa Sylow  $p$ -subgroups.
- a group  $G$  has property  $\overline{\mathcal{C}}_p$  if it satisfies  $\overline{\mathcal{Y}}_p$  and has Dedekind Sylow  $p$ -subgroups.

The relationships between  $\mathcal{Y}_p$  and  $\overline{\mathcal{Y}}_p$ ,  $\mathcal{X}_p$  and  $\overline{\mathcal{X}}_p$ , and  $\mathcal{C}_p$  and  $\overline{\mathcal{C}}_p$  is the content of the next two theorems.



## Theorem

Let  $p$  be a prime and  $G$  a group.

- If  $G$  is a  $\overline{\mathcal{Y}}_p$ -group, then it is a  $\mathcal{Y}_p$ -group. The converse is false.
- $G$  is a solvable  $\mathcal{PST}$ -group if and only if  $G$  is a  $\overline{\mathcal{Y}}_p$ -group for all primes  $p$ .

## Theorem

Let  $p$  be a prime and  $G$  a group.

- If  $G$  is an  $\overline{\mathcal{X}}_p$ -group ( $\overline{\mathcal{C}}_p$ -group). Then it is an  $\mathcal{X}_p$ -group ( $\mathcal{C}_p$ -group). The converse is false for  $\mathcal{X}_p$  ( $\mathcal{C}_p$ ).
- $G$  is a solvable  $\mathcal{PT}$ -group ( $\mathcal{T}$ -group) if and only if it is an  $\overline{\mathcal{X}}_p$ -group ( $\overline{\mathcal{C}}_p$ -group) for all primes  $p$ .

## Theorem

*Let  $G$  be solvable with  $D$  a system normalizer of  $G$ . Then  $G$  is a  $\mathcal{PST}$ -group if and only if  $D$  is a Hall subgroup with Hall complement  $L$  for which every subgroup of  $L$  permutes with  $D$ .*

## Proof.

- Assume  $G$  is a solvable  $\mathcal{PST}$ -group.
- The nilpotent residual, say  $L$ , is abelian and Hall.
- 9.2.7 of Robinson's text says a system normalizer, say  $D$ , complements  $L$ .
- Since  $G$  acts as power automorphisms on  $L$ ,  $D$  normalizes each subgroup of  $L$ .
- The desired result follows.



## Theorem

Let  $G$  be solvable with  $D$  a system normalizer of  $G$ . Then  $G$  is a  $\mathcal{PST}$ -group if and only if  $D$  is a Hall subgroup with Hall complement  $L$  for which every subgroup of  $L$  permutes with  $D$ .

## Proof.

- Assume  $D$  Hall in  $G$  with Hall complement  $L$  with all subgroups of  $L$  permuting with  $D$ .
- $G = LD = LD^g$  for all  $g \in G$ . Hence  $L$  permutes with all system normalizers of  $G$ .
- Hence  $L$  is subnormal and hence normal by B.-Bolinches, Cossey, and E.-Soler.
- For  $H \leq L$ ,  $L \trianglelefteq G$  implies  $H^{g^{-1}}D \leq G$  which implies  $HD^g \leq G$  for all  $g \in G$ .
- Hence every subgroup of  $L$  is subnormal in  $G$  and  $L$  is nilpotent.
- So  $G = LD$  is a mutually permutable product of nilpotent groups. Hence the nilpotent residual of  $G$  is abelian by B.-Bolinches, Cossey, and E.-Romero.
- $G/L \simeq DL/L$  is nilpotent so that  $L$  contains the nilpotent residual of  $G$ .
- By 9.2.7 of Robinson's text, we have that the nilpotent residual of  $G$  must complement  $D$  and must then be all of  $L$ .
- Since  $L$  is abelian and  $D$  acts as power automorphisms on  $L$ , we have that  $G = DL$  acts as power automorphisms on  $L$ . So  $G$  is  $\mathcal{PST}$  by Agrawal's result.



References are available upon request.  
Thank You!  
Questions?

