Finite Solvable Groups in which Semipermutability is a Transitive Relation

Matthew F. Ragland

Department of Mathematics
Auburn University Montgomery

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Joint work with:
Khaled Al-Sharo, Jim Beidleman and Hermann Heineken

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Permutability Concepts

All groups are finite.

- A subgroup H of a group G is said to *permute* with the subgroup K if HK is a subgroup of G.
- H is said to be *permutable (S-permutable)* in G if it permutes with all the subgroups (Sylow subgroups) of G.
- H is said to be *semipermutable (S-semipermutable)* in G if it permutes with all subgroups (Sylow subgroups) K of G for which $(|H|, |K|) = 1$.
- Subgroups H and K of G are said to be *mutually permutable* in G provided that H permutes with every subgroup of K and K permutes with every subgroup of H.
- H and K are said to be *totally permutable* in G if every subgroup of H permutes with every subgroup of K.

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Sylow Basis

- A set of pairwise permutable Sylow subgroups of a group G, one for each prime dividing the order of G, is called a *Sylow basis* of G.
- Let $\{p_1, p_2, \ldots, p_r\}$ be the set of prime divisors of $|G|$ and let $\{P_1, P_2, \ldots, P_r\}$ be a Sylow basis of G with $P_i \in \mathrm{Syl}_{p_i}(G)$ for all i.
- If π is a subset of $\{p_1, p_2, \ldots, p_r\}$, then $H = \prod_{p_i \in \pi} P_i$ is a Hall π-subgroup of G, that is $([G : H], |H|) = 1$.
- A second Sylow basis $\{Q_1, Q_2, \ldots Q_r\}$ is said to be conjugate to $\{P_1, P_2, \ldots P_r\}$ if there is an element $g \in G$ such that $P_i^g = Q_i, \ \ 1 \leq i \leq r$.

Results of Phillip Hall

Theorem (P. Hall)

- *A group G is solvable if and only if it has a Sylow basis.*
- *In a solvable group G any two Sylow bases are conjugate.*
- *In a solvable group G every π-subgroup, π a set of primes, is contained in a Hall π-subgroup. Moreover all the Hall π-subgroups of G are conjugate.*

- O. Kegel proved that an S-permutable subgroup is subnormal.
- We say that S-permutability is a transitive relation in a group G if when a subgroup H that is S-permutable in an S-permutable subgroup K of G is also S-permutable in G.
- We define what it means for semipermutability to be transitive in a similar way.
- A group G is called a $\mathcal{PST}$-group if S-permutability is a transitive relation in G. By Kegel's result $\mathcal{PST}$-groups are exactly those groups in which all subnormal subgroups are S-permutable.
- A group G is called a $\mathcal{BT}$-group ($\mathcal{SBT}$-group) if semipermutability (S-semipermutability) is a transitive relation

Solvable $\mathcal{PST}$-groups and $\mathcal{BT}$-groups

Theorem (Agrawal 1975)

A solvable group is a $\mathcal{PST}$-group if and only if it has a normal abelian Hall subgroup L such that G/L is nilpotent and G acts by conjugation on L as power automorphisms. (Note: The subgroup L may be taken to be the nilpotent residual of G.)

Theorem (Li, Wang and Wang 2008)

Let L denote the nilpotent residual of a group G. G is a solvable $\mathcal{BT}$-group if and only if G is a solvable $\mathcal{PST}$-group where $[P, Q] = 1$ for any $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ with p and q not dividing $|L|$.

- We note that the classes of finite solvable $\mathcal{BT}$ and $\mathcal{PST}$-groups are closed under forming quotients and subgroups.
- Also, the class of nilpotent groups is a subclass of the solvable $\mathcal{BT}$-groups.

Additional results on $\mathcal{BT}$-groups and $\mathcal{SBT}$-groups

Theorem (Li, Wang and Wang 2008)

The following are equivalent for a finite group G.

- *G is a solvable $\mathcal{BT}$-group;*
- *G is a solvable $\mathcal{SBT}$-group;*
- *every subgroup of G of prime power order is semipermutable in G;*
- *every subgroup of G is semipermutable in G;*
- *every subgroup of G is S-semipermutable in G;*
- *every subgroup of G of prime power order is S-semipermutable in G;*

Theorem (Kunren 2003)

G is a solvable $\mathcal{SBT}$-group if and only if G is a solvable $\mathcal{PST}$-group and every Sylow subgroup of G is S-semipermutable.

Two Motivating Questions

Question 1

What can one say about a group possessing a complete set of Sylow subgroups whose members are not only pairwise permutable, as in a Sylow basis, but are pairwise mutually permutable?

Question 2

What can one say about a group if every complete set of Sylow subgroups consists of subgroups whose members are mutually permutable?

Strong Sylow Bases

- Let $\{P_1, \dots, P_k\}$ be a Sylow basis for the solvable group G. Then $G = P_1 P_2 \cdots P_k$ is the product of the pairwise permutable subgroups P_1 , P_2 , . . . , P_k.
- If the subgroups P_1 , . . . , P_k are pairwise mutually permutable, then $\{P_1, \dots, P_k\}$ is called a *strong Sylow basis* for G.

Theorem (Asaad, Ballester-Bolinches, Beidleman, Esteban-Romero 2008)

Let $G = G_1 G_2 \cdots G_k$ be the product of pairwise mutually permutable solvable $\mathcal{PST}$-subgroups G_1 , G_2 , . . . , G_k . Then G is supersolvable. In particular, if G_1 , G_2 , . . . , G_k are nilpotent, then G is supersolvable.

- By the previous theorem, we see that a group with a strong Sylow basis is supersolvable.

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Strong Sylow Bases

Theorem

Let G be a group.

- *G is a solvable $\mathcal{PST}$-group if and only if it has a strong Sylow basis.*
- *If $\{P_1, \ldots, P_k\}$ is a strong Sylow basis for G, then the subgroups $P_1, \ldots, P_k$ are pairwise totally permutable.*
- *If G has a strong Sylow basis, then every Sylow basis is strong.*
- *If G has a strong Sylow basis, then so does every subgroup of G.*

The Connection with System Normalizers

Recall that a system normalizer associated with a Sylow basis of a solvable group is the intersection of all the normalizers of the Sylows in the basis.
Since we have characterized the solvable $\mathcal{PST}$-groups in terms of their Sylow bases and mutual permutability, it only seems natural that there is a way to characterize them in terms of their system normalizers and permutability.
The following is a truly remarkable result:

Theorem (Ballester-Bolinches, Cossey, Escriva-Soler 2009?)

Let G be a solvable group. If a subgroup H of G permutes with each system normalizer of G, then H is subnormal in G.

Theorem (Ballester-Bolinches, Cossey, Esteban-Romero 2005)

If $G = AB$ is the totally permutable product of the two nilpotent subgroups A and B, then G has an abelian nilpotent residual.

The Connection with System Normalizers

Theorem

Let G be solvable with D a system normalizer of G. G is a $\mathcal{PST}$-group if and only if D is a Hall subgroup with Hall complement L for which every subgroup of L permutes with D.

The following corollary is immediate.

Corollary

Let G be solvable with D a system normalizer of G. G is a $\mathcal{PST}$-group if and only if D is a Hall subgroup with Hall complement L where $G = LD$ is a mutually permutable product of L with D.

Relaxing the Permutability Conditions

Having a strong Sylow basis forces a lot of permutability conditions on the Sylow subgroups. It is reasonable to ask if some of these permutability conditions can be relaxed and still yield a $\mathcal{PST}$-group.

Definition

Let N denote a normal subgroup of G. Let us call $\Sigma = \{P_1, P_2 \ldots, P_k\}$ a strong Sylow system of G with respect to N if Σ consists of a set of Sylow subgroups of G, $P_i \in \mathrm{Syl}_{p_i}(G)$, where for each $P_i \in \Sigma$ with $p_i \in \pi(N)$ we have $P_i P_j$ is a mutually permutable product for all $P_j \in \Sigma$.

Theorem

Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N. The following statements are equivalent:

- *G has a strong Sylow basis,*
- *G has a strong Sylow system with respect to N,*
- *G is a solvable $\mathcal{PST}$-group.*

Is there connection between $\mathcal{BT}$-groups and Sylow Bases?

Theorem

Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N. The following statements are equivalent:

- *G has a strong Sylow basis,*
- *G has a strong Sylow system with respect to N,*
- *G is a solvable $\mathcal{PST}$-group.*

Theorem

Let G be a group and N a normal subgroup of G with nilpotent quotient group G/N. The following statements are equivalent:

- *Every complete set of Sylows of G is a strong Sylow basis,*
- *Every complete set of Sylows of G is a strong Sylow system of G with respect to N,*
- *G is a solvable $\mathcal{BT}$-group.*

Some Additional Results

Theorem

Let G be a group with normal Sylow p-subgroup P and normal Sylow q-subgroup Q. If G/P and G/Q are solvable $\mathcal{PST}$-groups, then G is a solvable $\mathcal{PST}$-groups.

Theorem

Let G be a group with normal Sylow p-subgroup P and normal Sylow q-subgroup Q. If G/P and G/Q are solvable $\mathcal{BT}$-groups, then G is a solvable $\mathcal{BT}$-groups.

Theorem

A group G is a solvable $\mathcal{PST}$-group if and only if it has a normal solvable $\mathcal{PST}$-subgroup N such that G/N'' is a solvable $\mathcal{PST}$-group.

Theorem

A group G is a solvable $\mathcal{BT}$-group if and only if it has a normal solvable $\mathcal{BT}$-subgroup N such that G/N'' is a solvable $\mathcal{BT}$-group.

A Local Characterization

Definition

A group G is called a Y_p-group if whenever K is a p-subgroup of G every subgroup of K is S-permutable in $N_G(K)$.

Theorem (Ballester-Bolinches and Esteban-Romero 2002)

A group G is a solvable $\mathcal{PST}$-group if and only if G is a Y_p-group for all primes p.

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More Local Characterizations

Definition

- A group G is a $\widehat{Y}_p$*-group* if for every p-subgroup K of G every subgroup of K is semipermutable in $N_G(K)$.
- A group G is a $\widetilde{Y}_p$*-group* if for every p-subgroup K of G every subgroup of K is S-semipermutable in $N_G(K)$.
- A group G is a $\overline{Y}_p$*-group* if for every p-subgroup K of G, $N_G(K)$ has a strong Sylow basis.
- A group G is a $\overline{\overline{Y}}_p$*-group* if for every p-subgroup K of G, every complete set of Sylow subgroups of $N_G(K)$ is a strong Sylow basis.

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More Local Characterizations

Theorem

Let p be a prime then $\overline{\overline{Y}}_p \subsetneq \overline{Y}_p \subsetneq \widehat{Y}_p = \widetilde{Y}_p = Y_p$.

Corollary

Let G be a group. Then the following are equivalent:

- *G is a solvable* $\mathcal{PST}$*-group.*
- *G is a* $\widetilde{Y}_p$*-group for all primes p.*
- *G is a* $\widehat{Y}_p$*-group for all primes p.*

Theorem

Let G be a group.

- *G is a solvable* $\mathcal{BT}$*-group if and only if G is a* $\overline{\overline{Y}}_p$*-group for all primes p.*
- *G is a solvable* $\mathcal{PST}$*-group if and only if G is a* $\overline{Y}_p$*-group for all primes p.*

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References are available upon request. Thank you! Questions?