Groups with a Finite Covering by Isomorphic Abelian Subgroups

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Introduction

What is a *CIA*-group?

A group G has a finite covering by proper isomorphic abelian subgroups if $G = \bigcup_{i=1}^{n} A_i$ where the A_i's are proper isomorphic abelian subgroups of G. We will call such groups *CIA*-groups.

Outline

- Examples of *CIA*-groups.
- Structure of abelian *CIA*-groups.
- Structure of nilpotent *CIA*-groups.
- Non-*CIA*-groups.
- Every group is a direct factor of a *CIA*-group.
- Do all *CIA*-groups have nontrivial center?

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Examples

- Cyclic groups are not *CIA*-groups.
- The Klein four group is a *CIA*-group.
- $C_4 \times C_2$ is not a *CIA*-group.
- The quaternion group Q_8 is a *CIA*-group.
- C_6 is not a *CIA*-group.
- $C_6 \times C_2$ is a *CIA*-group.
- $C_6 \times C_4$ is not a *CIA*-group.
- A_5 is not a *CIA*-group.
- $A_5 \times C_{30}$ is a *CIA*-group.
- S_4 is not a *CIA*-group.
- $S_4 \times C_6 \times C_4$ is a *CIA*-group.
- If G is a nonabelian group of $\exp(G) = p^2$ and $|G| = p^3$ where p is an odd prime, then G is not a *CIA*-group.

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Two Lemmas

Lemma 1

Let H be a CIA-group. Suppose K is either a CIA-group or an abelian group. Then $H \times K$ is a CIA-group.

Lemma 2

Let $P = P_1 \times P_2 \times \cdots \times P_t$ be homocyclic where each P_i is a cyclic group of order p^α. Then P is the union of subgroups isomorphic to C_{p^α}. Moreover, if $t > 1$, then P is a CIA-group.

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Structure of Finite Abelian *CIA*-Groups of Prime Power Order

Theorem 1

Let P be an abelian p-group with order p^β. Write $P = P_1 \times \cdots \times P_t$ with $P_i = C^1_{p^{\beta_i}} \times \cdots \times C^{n_i}_{p^{\beta_i}}$ where $C^k_{p^{\beta_i}} \simeq C_{p^{\beta_i}}$ for all k and i, $\beta_i \neq \beta_j$ for $i \neq j$, and $\sum_{i=1}^t n_i\beta_i = \beta$. Then the following hold:

1. P can be written as the union of subgroups isomorphic to $C_{p^{\beta_1}} \times \cdots \times C_{p^{\beta_t}}$.
2. P is a *CIA*-group if and only if $n_i > 1$ for some i.

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Structure of Finite Abelian CIA-Groups

Theorem 2

Let G be a finite abelian group. G is a CIA-group if and only if some Sylow subgroup of G is a CIA-group.

Corollary 1

Let G be a finite abelian group. G is a CIA-group if and only if there exists a direct factor of G with the form $C_{p^n} \times C_{p^n}$ for some prime p and some integer n.

Corollary 2

A finite cyclic group is not a CIA-group.

Maximum root

Given a finite group G, $g \in G$ is a maximum root of G provided whenever $h^n = g$ we have $|g| = |h|$. We denote by

$$\pi_{me}(G) = \{|g| : g \text{ is a maximum root in } G\},$$

and $C_{\pi_{me}(G)} = \prod_{n \in \pi_{me}(G)} C_n$.

Corollary 3

Let G be a finite abelian group. G is CIA-group if and only if $P \not\simeq C_{\pi_{me}(P)}$ for some Sylow subgroup P of G.

Corollary 4

Let G be a finite abelian CIA-group with Sylow basis Σ and let $\{A_i\}_{i=1}^{n}$ be a collection of isomorphic abelian subgroups of G. Then $\{A_i\}_{i=1}^{n}$ is a covering of G if and only if

$$A_i \simeq \left(\prod_{P \in \Sigma} C_{\pi_{me}(P)} \right) \times H$$

where H is any subgroup of G trivially intersecting and not complementing $\left(\prod_{P\in\Sigma} C_{\pi_{me}(P)}\right)$ in G.

Theorem 3

Let G be a finite nilpotent group. Write $G = P_1 \times \cdots \times P_n$ where $P_i \in \mathrm{Syl}_{p_i}(G)$. Then G is a *CIA*-group if and only if G satisfies the following two conditions:

1. Every Sylow subgroup of G is abelian or is a *CIA*-group.
2. At least one Sylow subgroup of G is a *CIA*-group.

Corollary 5

Every finite Hamiltonian group is a *CIA*-group.

Non-*CIA*-groups

None of the following classes of groups are *CIA*-groups.

- Cyclic Groups
- Groups of square-free order
- Dihedral groups (D_{2n}, $n \geq 3$)
- Frobenius groups
- Symmetric groups (S_n, $n \geq 2$)
- Alternating groups (A_n, $n \geq 3$)
- Simple groups

Every Group is a Direct Factor of a *CIA*-Group

Definition

For a finite group G, define $\pi_e(G)$ and $C_{\pi_e(G)}$ by:

$$\pi_e(G) = \{|a| : a \in G\} \text{ and } C_{\pi_e(G)} = \prod_{n \in \pi_e(G)} C_n$$

Theorem 4

If G is a finite group, then $G \times C_{\pi_e(G)}$ is a *CIA*-group covered by abelian subgroups isomorphic to $C_{\pi_e(G)}$.

Corollary 6

Every finite group is a direct factor of a *CIA*-group.

A Proof of Theorem 4

Theorem 4

If G is a finite group, then $G \times C_{\pi_e(G)}$ is a CIA-group covered by abelian subgroups isomorphic to $C_{\pi_e(G)}$.

Proof.

- Write $\pi_e(G) = \{n_1, n_2, \ldots, n_t\}$.
- Write $H = G \times C_{n_1} \times C_{n_2} \times \cdots \times C_{n_t}$.
- Let $\pi_0 : H \to G$ be the natural projection of H onto G.
- Let $\pi_k : H \to C_{n_k}$ be the natural projection of H onto C_{n_k}.
- Let $h \in H$ and let $n_k = |\pi_0(h)|$.
- Let $A = C_{n_1} \times \cdots \times C_{n_{k-1}} \times \langle \pi_0(h)\pi_k(h) \rangle \times C_{n_{k+1}} \times \cdots \times C_{n_t}$.
- Then $A \simeq C_{\pi_e}(G)$ and $h \in A$.
- Deduce G is covered by copies of $C_{\pi_e(G)}$.

□

Centerless *CIA*-groups?

Example

There is a group of order 216 isomorphic to the semidirect product $((C_3 \times C_3) \rtimes Q_8) \rtimes C_3$ which has center of order two.

Open Question

Do all *CIA*-groups have nontrivial center?

Questions