Groups in which the Hypercentral Factor Group has a Transitive Normality Relation

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Definitions

What are $\mathcal{T}$-groups $\mathcal{PT}$-groups and $\mathcal{PST}$-groups?

- A $\mathcal{T}$-group is a group in which normality is a transitive relation.
- A $\mathcal{PT}$-group is a group in which permutability is a transitive relation.
- A $\mathcal{PST}$-group is a group in which Sylow-permutability is a transitive relation.

Or Equivalently...

- A $\mathcal{T}$-group is a group in which every subnormal subgroup is normal.
- A $\mathcal{PT}$-group is a group in which every subnormal subgroup is permutable.
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Theorems of Gaschütz, Zacher, and Agrawal

Theorem [Gaschütz 1957]

G is a solvable $\mathcal{T}$-group if and only if the following hold for G:

- $\gamma_*(G)$ is an abelian Hall subgroup of G of odd order;
- G acts as power automorphisms on $\gamma_*(G)$;
- $G/\gamma_*(G)$ is a $\mathcal{T}$-group.

Theorem [Zacher 1964]

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Theorem [Agrawal 1975]

G is a solvable $\mathcal{PST}$-group if and only if the following hold for G:

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$\mathcal{T}$-, $\mathcal{PT}$-, and $\mathcal{PST}$-groups Only Differ in their Sylows

Theorem [Agrawal 1975]

- G is a solvable $\mathcal{PST}$-group with Dedekind Sylow subgroups if and only if G is a solvable $\mathcal{T}$-group.
- G is a solvable $\mathcal{PST}$-group with Iwasawa (modular) Sylow subgroups if and only if G is a solvable $\mathcal{PT}$-group.

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$\mathcal{T}_1$-groups and $\mathcal{T}_0$-groups

Definition

- Let $\mathcal{T}_1$ denote the class of groups G where $G/Z_*(G)$ is a $\mathcal{T}$-group.
- Let $\mathcal{T}_0$ denote the class of groups G where $G/\Phi(G)$ is a $\mathcal{T}$-group.

We are interested in characterizing $\mathcal{T}_1$ and $\mathcal{T}_0$ groups in terms of the already mentioned theorems for $\mathcal{T}$-groups. Also, one can define the classes $\mathcal{PT}_1$, $\mathcal{PT}_0$, $\mathcal{PST}_1$, $\mathcal{PST}_0$ in a manner similar to $\mathcal{T}_1$ and $\mathcal{T}_0$. We are interested in the structure of these classes as well.

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Basic Lemmas on $\mathcal{T}_1$ and $\mathcal{T}_0$

Lemma [Beidleman and Heineken 2006]

Let G be a solvable $\mathcal{T}_1$-group. Then

- G is supersolvable.
- If $H \trianglelefteq G$, then G/H is a $\mathcal{T}_1$-group.
- If H is a subgroup of G, then H is a $\mathcal{T}_1$-group.

Lemma [Fransman and Van der Waall 1996]

- A solvable $\mathcal{T}_0$-group supersolvable.
- If $H \trianglelefteq G$ with G a $\mathcal{T}_0$-group, then G/H is a $\mathcal{T}_0$-group.

Recent results on $\mathcal{T}_1$-groups

Theorem [Beidleman and Heineken 2006]

For a solvable group G, the following are equivalent.

- G is a $\mathcal{PST}$-group.
- G is a $\mathcal{T}_1$-group with $(|\gamma_*(G)|, |Z_*(G)|) = 1$.

Theorem [Beidleman and Heineken 2006]

Let G be a solvable $\mathcal{T}_1$-group. Then $\gamma_*(G)$ is nilpotent of class at most 2 and G acts by conjugation on $\gamma_*(G)/\gamma_*(G)'$ as a group of power automorphisms.

Theorem [Beidleman, Heineken, and Ragland 2007]

G is a solvable $\mathcal{T}_1$-group if and only if the following hold:
(i) $\gamma_*(G)Z_*(G)/Z_*(G)$ is an abelian Hall subgroup of $G/Z_*(G)$
(ii) G acts by conjugation on $\gamma_*(G)/\gamma_*(G)'$ as a group of power automorphisms.

Recent results on $\mathcal{T}_1$-groups

Theorem [Fransman and Van der Waall 1996]

The following are equivalent:

- G is a solvable $\mathcal{T}$-group.
- G is a subgroup-closed $\mathcal{T}$-group.
- G is a supersolvable $\mathcal{T}$-group.

Theorem [Beidleman, Heineken, and Ragland 2007]

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What about subgroup-closure for $\mathcal{T}_0$?

Theorem [Ballester-Bolinches, Esteban-Romero, and Pedraza-Aguilera 2005; Ragland 2005]

The following are equivalent:

- G is a subgroup-closed $\mathcal{T}_0$-group.
- G is a solvable $\mathcal{PST}$-group.

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Lemma [Ragland 2005]

For a group G, if $G/\Phi(G)$ is a solvable $\mathcal{PST}$-group, then G is a $\mathcal{T}_0$-group.

Theorem [Ragland 2005]

The classes of solvable $\mathcal{T}_0$-groups, $\mathcal{PT}_0$-groups, and $\mathcal{PST}_0$-groups are all equal.

Lemma [Beidleman, Heineken, and Ragland 2007]

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Existence of Hall subgroups in $\mathcal{T}_0$-groups

Lemma [Ragland 2005]

If G is a solvable $\mathcal{T}_0$-group, then $\gamma_*(G)$ is a nilpotent Hall subgroup of G of odd order.

Theorem [Ragland 2005]

G is a solvable $\mathcal{T}_0$-group if and only if $\gamma_*(G)$ is a nilpotent Hall subgroup of G of odd order and G acts by conjugation as a group of power automorphisms on $\gamma_*(G)\Phi(G)/\Phi(G)$.

Theorem [Agrawal 1975]

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- $\gamma_*(G)$ is an abelian Hall subgroup of G of odd order;
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Existence of Hall subgroups in $\mathcal{T}_1$-groups

Lemma [Beidleman, Heineken, and Ragland 2007]

If G is a solvable $\mathcal{T}_1$-group with H the Hall π-subgroup of Fit(G) where π is set of primes dividing the order of $\gamma_*(G)$, then H is a nilpotent Hall subgroup of G of odd order.

Theorem [Beidleman, Heineken, and Ragland 2007]

G is a solvable $\mathcal{T}_1$-group if and only if H is a nilpotent Hall subgroup of G of odd order and G acts by conjugation as a group of power automorphisms on $HZ_*(G)/Z_*(G)$. (Here H is as in the previous Lemma.)

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Definitions

- Let $\mathscr{C}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have $H \trianglelefteq N_G(P)$.
- Let $\mathscr{X}_p$ be the class of groups G satisfying the property that, whenever $H \leq P \in \mathrm{Syl}_p(G)$, we have H per $N_G(P)$.
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Local Theorems for $\mathcal{T}$, $\mathcal{PT}$, and $\mathcal{PST}$

Theorem [Robinson 1968]

$G \in \mathscr{C}_p$ for all primes p if and only if G is a solvable $\mathcal{T}$-group.

Theorem [Beidleman, Brewster, Robinson 1999]

$G \in \mathscr{X}_p$ for all primes p if and only if G is a solvable $\mathcal{PT}$-group.

Theorem [Ballester-Bolinches, Esteban-Romero 2002]

$G \in \mathscr{Y}_p$ for all primes p if and only if G is a solvable $\mathcal{PST}$-group.

Theorem [Ballester-Bolinches, Esteban-Romero 2002]

A group G is an $\mathscr{X}_p$-group (respectively, $\mathscr{C}_p$-group) if and only if G is a $\mathscr{Y}_p$-group and the Sylow p-subgroups of G are Iwasawa (respectively, Dedekind) groups.

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Theorem [Ragland 2005]

Let G be a group with $\Phi(G)_p$ the Sylow p-subgroup of $\Phi(G)$, $F = \mathrm{Fit}(G)$, F'_p the Sylow p-subgroup of F', $L = \gamma_*(G)$, and L'_p the Sylow p-subgroup of L'. Then following are equivalent:

- G is a solvable $\mathcal{T}_0$-group.
- $G/\Phi(G)_p$ is a $\mathscr{Y}_p$-group for all primes p.
- $G/\Phi(G)_p$ is a $\mathscr{X}_p$-group for all primes p.
- $G/\Phi(G)_p$ is a $\mathscr{C}_p$-group for all primes p.
- G/F'_p is a $\mathscr{Y}_p$ group for all primes p.
- L is nilpotent and G/L'_p is a $\mathscr{Y}_p$ group for all primes p.

A Local Approach to $\mathcal{T}_1$

Theorem [Beidleman, Heineken, and Ragland 2007]

Let G be a group with Z_p the Sylow p-subgroup of $Z_*(G)$ Then following are equivalent:

- G is a solvable $\mathcal{T}_1$-group.
- G/Z_p is a $\mathscr{Y}_p$-group for all primes p.
- G/Z_p is a $\mathscr{X}_p$-group for all primes p.
- G/Z_p is a $\mathscr{C}_p$-group for all primes p.

We believe there are similar local results where one considers the quotient by the Sylow p-subgroup of H where H is the Hall π-subgroup of the Fitting subgroup where $\pi = \pi(\gamma_*(G))$. We plan to explore many other locally defined classes in the near future.

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References

1. R.K. Agrawal, *Finite groups whose subnormal subgroups permute with all Sylow subgroups*, Proc. Amer. Math. Soc. **47** (1975), 77-83.

2. A. Ballester-Bolinches and R. Esteban-Romero, *Sylow permutable subnormal subgroups of finite groups* J. Algebra **251** (2002), 727-738.

3. A. Ballester-Bolinches, R. Esteban-Romero and M.C. Pedraza-Aguilera, *On a class of p-soluble groups* Algebra Colloquium **12** (2005), 263-267.

4. J. Beidleman, B. Brewster, and D.J.S. Robinson, *Criteria for permutability to be transitive in finite groups*, J. Algebra **222** (1999), 400-412.

References

5. J.C. Beidleman and H. Heineken, *Groups in which the hypercentral factor group is a T-group*, Ricerche di Matematica **55**(2006), 219-225.

6. W. Gaschütz, *Gruppen, in denen das Normalteilersein transitiv ist.* J. Reine u. Angew. Math. **198** (1957), 87-92.

7. M.F. Ragland, *Generalizations of Groups in which Normality is transitive*, Comm. in Algebra **35** (2007), 3242-3252.

8. M.F. Ragland, *On Generalizations of Finite Groups in which Normality is a Transitive Relation*, Ph-D Thesis, University of Kentucky, 2005.

References

9. D.J.S. Robinson, *A note on finite groups in which normality is transitive*, Proc. Amer. Math. Soc. **19** (1968), 933-937.

10. R.W. Van der Waall and A. Fransman, *On products of groups for which normality is a transitive relation on their Frattini factor groups*, Quaestiones Math. **19** (1996), no. 1-2, 59-82.

11. G. Zacher, *I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogruppi quasi-normali*, Atti Accad. Naz.Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **37**(1964), 150-154.