JOURNAL OF ALGEBRA 165, 633–635 (1994)

On Seminormal Subgroups

TUVAL FOGUEL*,†

Mathematical Science Research Institute, 1000 Centennial Drive, Berkeley, California 94720

Communicated by Walter Feit

Received September 28, 1992

In this paper we classify which simple groups contain proper nontrivial seminormal subgroups. We also develop a Suchr–Zassenhauz like theorem for seminormal subgroups. © 1994 Academic Press, Inc.

INTRODUCTION

In [6] Su introduces the concept of seminormal subgroups and using this tool he gives four sufficient conditions for supersolvability. In [7] Wang uses seminormal subgroups in order to obtain some sufficient conditions for nilpotency. Since every normal subgroup is seminormal in this paper we classify which simple groups contain proper nontrivial seminormal subgroups. This classification gives examples of seminormal subgroups that are not quasinormal, since by [5, 13.2.1] any quasinormal subgroup is subnormal. We also develop a Suchr–Zassenhauz like theorem for seminormal subgroups. In this paper all groups are assumed to be finite.

For the reader's convenience we include the following three theorems:

THEOREM [7, Theorem 1; 6, Proposition 1]. *If A is seminormal in G, then for each x in G, A^x is also seminormal and $S(A) = S(A^x)$. Moreover, if $B \in S(A)$, then for each y in G, $B^y \in S(A)$.*

THEOREM [7, Theorem 3; 6, Proposition 3]. *Let M be a maximal subgroup of G, then M is seminormal in G if and only if the index of M in G is a prime.*

THEOREM [5, Satz 3]. *Let A and B be subgroups of G. Suppose that for all $x \in G$, A and B^x are permutable; if $AB \neq G$, then at least one of A and B is contained in a proper normal subgroup of G.*

* Research at MSRI supported in part by NSF Grant DM 9022140.

† Department of Mathematics, University of the West Indies, Mona, Kingston 7, Jamaica, West Indies.

0021-8693/94 $6.00

Seminormal Subgroups of Simple Groups

THEOREM 1. *A simple group G contains a proper nontrivial seminormal subgroup A if and only if the index of A in G is a prime.*

Proof. If A is a subgroup of prime index in G then by [7, Theorem 3], $A <_{sn} G$.

Assume A is a proper nontrivial seminormal subgroup of G, and that $[G : A] = rs$ where r is a prime and $s \neq 1$. Let $B \in S(A)$, there exists a $b \in B$ such that $o(b) = r$. Then, for $g \in G$, $A\langle b\rangle^g = \langle b\rangle^g A < G$ since $b^g \in B^g \in S(A)$ [7, Theorem 1]. Thus by [5, Satz 3], G contains a proper normal subgroup, a contradiction. ∎

LEMMA 1. *If G is a simple group and G has a subgroup of prime index p, then p is the largest prime dividing $|G|$ and p^2 does not divide $|G|$, and $p > 3$.*

Proof. Since G contains a subgroup of prime index, we get a map from G to Σ_p. And since G is simple $|G| \mid |\Sigma_P|$. ∎

THEOREM 2. *A simple group G contains a proper nontrivial seminormal subgroup if and only if one of the following holds:*

(a) *$G = A_p$ and $H \cong A_{p-1}$ (p a prime).*

(b) *$G = PSL_n(q)$ (q a prime) and H is the stabilizer of a line or hyperplane, and $[G : H] = (q^n - 1)/(q - 1) = p$, p a prime (note in this case n is a prime greater than 2).*

(c) *$G = PSL_2(11)$ and $H \cong A_5$, in this case $[G : H] = 11$.*

(d) *$G = M_{23}$ and $H \cong M_{22}$ or $G = M_{11}$ and $H \cong M_{10}$.*

Proof. From [3, Theorem 1] we see that the above are the only options for a simple group G to have a subgroup of prime index. Note that in (b), $n > 2$ since $p > 3$. ∎

THEOREM 3. *If G is a simple group, then all proper nontrivial seminormal subgroups of G are conjugate in* Aut(G).

Proof. Lemma 1 gives the uniqueness of p, thus by [3, Theorem 1; 1] all proper nontrivial seminormal subgroups of a simple group are conjugate in Aut(G). ∎

Suchr–Zassenhauz for Seminormal Subgroups

LEMMA 2. *If $A <_{sn} G$ then for any $B \in S(A)$ if $X = \{b \in B \mid b$ is a $\pi(A)$-element$\}$, $X^G \leqslant A$.*

Proof. For any $b \in X$, $A\langle b\rangle \leqslant G$ and $|A\langle b\rangle| = |A|\,|\langle b\rangle|/|\langle b\rangle|/|A \cap \langle b\rangle| = |A|$ since $|\langle b\rangle|$ is a $\pi(A)$-number.

Thus $\langle b\rangle \leqslant A$ and $T = \langle X\rangle \leqslant A$. Since for all $g \in G$, $B^g \in S(A)$ [7, Theorem 1] we see that $X^G = T^G \leqslant G$. ∎

THEOREM 4 (Suchr–Zassenhauz for Seminormal Subgroups). *Let A be a seminormal Hall π-subgroup of G, then any $B \in S(A)$ is a Hall π'-subgroup of G. In particular Hall π'-subgroups of G exist. And A commutes with all q-subgroups Q of G where q is a π'-prime.*

Proof. Given any $B \in S(A)$, set $K = B \cap A$, then since K is a π-subgroup, $K \leqslant A$ (Lemma 2). Since for every $b \in B$, and $k \in K$, $k^b \in B$ and is a π-element $k^b \in K$ and K is normal in $KB = B$.

Given that $B = KB \leqslant G$, since $[G : A] = [B : K]$, $(|K|, [B : K]) = 1$. And since K is normal in M, by the Suchr–Zassenhauz Theorem [5, Theorem 9.12] there exists a subgroup $T \leqslant B$ of order $[G : A]$, so $AT = G$ and $T = B$. Thus B is a Hall π'-subgroup of G.

Given a q-subgroup Q of G where q is a π'-prime, then $Q \in B^g \in S(A)$ [7, Theorem 1] for some $g \in G$, thus $AQ = QA$. ∎

Note. We can have two subgroups A and B of a group G such that $|A| = [G : B]$ and $(|B|, [G : B]) = 1$, but neither of them is seminormal (note that they are permutable by [4, Theorem 8]).

EXAMPLE. Let $G = PSL_3(2)$, then G has a subgroup A of index 8 and a sylow 2 subgroup of order 8, but from Theorem 2 neither is seminormal.

LEMMA 3. *If $A = \langle x\rangle <_{sn} G$ and $|A|$ is a prime, then either $A \leqslant \operatorname{Frat} G$ or for any $B \in S(A)$, $A \cap B = 1$.*

Proof. If $G \in S(A)$, then for all maximum subgroups M of G, $AM = MA = M$, so $A \leqslant \operatorname{Frat} G$. ∎

REFERENCES

1. J. H. CONWAY, R. T. CURTIS, S. P. NORTON, R. A. PARKER, AND R. A. WILSON, "Atlas of Finite Groups," Clarendon Press, Oxford, 1985.
2. O. H. KEGEL, Produktie nilpotenter Gruppen, *Arch. Math.* **12** (1961), 90–93.
3. R. M. GURALNICK, Subgroups of prime index in simple group, *J. Algebra* **81** (1983), 304–311.
4. O. ORE, Contribution to the theory of groups of finite order, *Duke Math. J.* **5** (1939), 431–438.
5. D. J. S. ROBINSON, "A Course in the Theory of Groups," Springer-Verlag, New York/Heidelberg/Berlin, 1982.
6. X. SU, Seminormal subgroups of finite groups, *J. Math. (Wuhan)* **8**, No. 1 (1988), 5–10.
7. P. WANG, Some sufficient conditions of a nilpotent group, *J. Algebra* **148** (1992), 289–295.