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On loops covered by subloops

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Dedicated to the memory of Bernhard Neumann

Abstract

The topic of this paper is the investigation of coverings of a loop by subloops. A loop has a covering by subloops if it is the set-theoretic union of proper subloops. If the set of subloops is finite, the covering is called finite. Coverings of groups by subgroups have been widely investigated and key results are detailed in the introduction. Various analogues for loops of the results for groups are obtained. An example of an infinite loop which is the union of three proper commutative subloops, but has no finite homomorphic images and has a trivial center, shows that the results for loops cannot be as general as for groups, justifying additional assumptions on the loops or the coverings.

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1. Introduction

A group is said to have a covering by subgroups if it is the set-theoretic union of proper subgroups, and if the set of subgroups is finite, we say the covering is finite. Such coverings have been widely studied in groups, and recently analogous problems for rings and

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for semigroups were discussed in [1,11], respectively. It is only natural to look into such problems in the more general setting of loops. This is the topic of this paper. (For basic facts on loops we refer to Section 2). To develop our theme, we first look at the background and history of group coverings.

Results on finite coverings by subgroups first appeared in a book by Scorza [17] with an emphasis on coverings by a small number of subgroups. The following theorem, rediscovered by other authors, e.g. [7,9], appeared for the first time in [17].

Theorem 1.1. *A group is the set-theoretic union of three proper subgroups if and only if the group has a homomorphic image isomorphic to the Klein* 4*-group*.

Bernhard Neumann in [12,13] investigated coverings by cosets. The following theorem, often called Neumann's Lemma, is a key to many of the group theoretic results. In particular, a characterization of groups having finite coverings is stated as a corollary of the following theorem.

Theorem 1.2. *Let* $G = \bigcup_{i=1}^{k} g_i H_i$, *where* $H_1, \ldots, H_k$ *are (not necessarily distinct) subgroups of* G. *Then, if we omit from the union any cosets* $g_i H_i$ *for which* $[G : H_i]$ *is infinite, the union of the remaining cosets is still all of* G.

Corollary 1.3. *A group has a finite covering by subgroups if and only if it has a finite non-cyclic homomorphic image*.

An unpublished result by Reinhold Baer (see [16, Theorem 4.6]) leads to the investigation of finite coverings by special subgroups as can be found in [5,10].

Theorem 1.4. *A group is central-by-finite if and only if it is the union of finitely many abelian subgroups*.

As in the case of semigroups in [11], we cannot expect results for loops as general as those stated above for groups. As we will show, the only result which carries over directly to loops is the well-known observation that no group is the union of two proper subgroups. In Section 4, we will provide an example of an infinite non-commutative loop which is the union of three proper subloops, each of them in fact is an abelian group. Furthermore, the loop has no finite homomorphic image, in particular none which is isomorphic to the Klein 4-group, and its center is trivial. On the other hand, given any of these three subloops, the loop can be written as the union of three cosets of this subloop.

Thus the question arises, as to what conditions we have to impose on the loop or its covering to obtain results analogous to those for groups as stated above. These results come in two stages. In the first, the assumptions of the Neumann Lemma for loops are not necessarily satisfied; in the second, they are.

In the first stage we require that the cosets for each of the subloops in the covering form a partition of the loop. This leads to a Poincaré theorem for loops, discussed in Section 3. With the help of this result we obtain in Section 5 several partial analogues of the results for groups, provided all subgroups in the covering have a finite index in the loop.

In the last section, we establish Neumann's Lemma for loops. Here, additional conditions have to be imposed on the coset decomposition of the loop modulo each subloop in the covering. Such a decomposition is called a strong coset decomposition, which can be best described as a closure operation on cosets. We establish another set of partial analogues of the group results, where this time the finite coverings satisfy the assumptions of Neumann's Lemma for loops.

Following common practice, we call a subloop or a normal subloop of a loop a subgroup or normal subgroup, respectively, provided it is a group. Coverings of loops by subgroups make for an interesting question in this context. As we will see in Section 5, various types of coverings by subgroups lead to different kinds of associativity conditions for the loop.

In conclusion, we want to mention that there are many other results on coverings of groups by subgroups which seem to be worth investigation in the setting of loops. Examples include coverings by normal subgroups [4] and by subgroups with a certain property, e.g. as in [5,10]. These investigations will be the topic of a future publication.

2. Preliminaries

In this section, we review a few concepts from loop theory, and establish some conventions concerning notation. For basic facts about loops and quasigroups, we refer the reader to Belousov [2], Bruck [6] and Pflugfelder [15].

A *magma* $\mathscr{L}$ consists of a set $\mathscr{L}$ together with a binary operation on $\mathscr{L}$. For $x \in \mathscr{L}$, define the left (right) translation by x as $L(x)y = xy$ $(R(x)y = yx)$ for all $y \in \mathscr{L}$. A magma in which all left and right translations are bijections is called a *quasigroup*. A quasigroup $\mathscr{L}$ is an *idempotent quasigroup* if for any $x \in \mathscr{L}$, $xx = x$. A quasigroup $\mathscr{L}$ with a two-sided identity element 1 such that for any $x \in \mathscr{L}$, we have $x1 = 1x = x$, is called a *loop*.

A loop $\mathscr{L}$ is called *power-associative*, if for any $x \in \mathscr{L}$, the subloop generated by x is a group. A loop $\mathscr{L}$ is *left power-alternative*, if for all $x, y \in \mathscr{L}$ $x^m(x^n y) = x^{m+n}y$ for any integers m and n. Similarly, we define a *right power-alternative loop*. A loop $\mathscr{L}$ is a *power-alternative loop*, if it is both a right and left power-alternative loop. Taking $n = -1$ and $m = 1$, we obtain the *left inverse property* (LIP) $x^{-1}(xy) = y$, and similarly the *right inverse property* (RIP) $(yx)x^{-1} = y$. A loop $\mathscr{L}$ has the *inverse property* (IP), if it has both the right and left inverse property. A loop $\mathscr{L}$ is *diassociative*, if for any $x, y \in \mathscr{L}$, the subloop generated by x and y is a group. Note that a diassociative loop has the inverse property.

The *left*, *middle*, and *right nucleus* of a loop $\mathscr{L}$ are defined, respectively, as

$$\mathrm{Nuc_l}(\mathscr{L}) := \{x \in \mathscr{L} : x(yz) = (xy)z \quad \forall y, z \in \mathscr{L}\},$$

$$\mathrm{Nuc_m}(\mathscr{L}) := \{y \in \mathscr{L} : x(yz) = (xy)z \quad \forall x, z \in \mathscr{L}\},$$

$$\mathrm{Nuc_r}(\mathscr{L}) := \{z \in \mathscr{L} : x(yz) = (xy)z \quad \forall x, y \in \mathscr{L}\}.$$

The nucleus of a loop $\mathscr{L}$ is defined as

$$\mathrm{Nuc}(\mathscr{L}) := \mathrm{Nuc_l}(\mathscr{L}) \cap \mathrm{Nuc_m}(\mathscr{L}) \cap \mathrm{Nuc_r}(\mathscr{L}).$$

Each of these is an associative subloop of $\mathscr{L}$, as follows from Theorem I.3.5 in [15]. The *centrum* and *center* of a loop $\mathscr{L}$ are defined, respectively, by

$$\begin{aligned} C(\mathscr{L}) &:= \{x \in \mathscr{L} : xy = yx \quad \forall y \in \mathscr{L}\}, \\ Z(\mathscr{L}) &:= \mathrm{Nuc}(\mathscr{L}) \cap C(\mathscr{L}). \end{aligned}$$

Given a loop $\mathscr{L}$, a subloop $\mathscr{K}$ is said to be *normal* if, for all $x, y \in \mathscr{L}$, $x(y\mathscr{K}) = (xy)\mathscr{K}$, $x\mathscr{K} = \mathscr{K}x$, and $(\mathscr{K}x)y = \mathscr{K}(xy)$ [6, p. 60, IV.1]. These three conditions are clearly equivalent to the pair $x(\mathscr{K}y) = \mathscr{K}(xy)$ and $x(\mathscr{K}y) = (x\mathscr{K})y$ for all $x, y \in \mathscr{K}$. Note that the center of a loop is a normal subloop. However, the centrum is not necessarily a subloop.

3. The index of a subloop and coset decomposition

In this section, we explore the notions of coset decomposition and index in the case of loops, leading to Poincaré's Theorem for loops.

Definition 3.1. Given a loop $\mathscr{L}$ and a subloop $\mathscr{H}$, then a *left coset* of $\mathscr{H}$ is a set of the form $x\mathscr{H} = \{xh : h \in \mathscr{H}\}$ where $x \in \mathscr{L}$, and a right coset has the form $\mathscr{H}x = \{hx : h \in \mathscr{H}\}$.

To avoid cumbersome repetitions, we state our results only for left cosets. They hold also for right cosets, unless stated otherwise.

Lemma 3.2. *If $\mathscr{H}$ and $\mathscr{K}$ are subloops of $\mathscr{L}$, then for $x \in \mathscr{L}$, $x(\mathscr{H} \cap \mathscr{K}) = x\mathscr{H} \cap x\mathscr{K}$.*

Proof. Clearly $x(\mathscr{H} \cap \mathscr{K}) \subseteq x\mathscr{H} \cap x\mathscr{K}$. Let $l \in x\mathscr{H} \cap x\mathscr{K}$, then $l = xh = xk$ for $h \in \mathscr{H}$ and $k \in \mathscr{K}$. By cancellation we obtain $h = k \in \mathscr{H} \cap \mathscr{K}$. ☐

The cosets of a subloop do not necessarily form a partition of the loop. This leads to the following definition.

Definition 3.3. A loop $\mathscr{L}$ has a left (right) *coset decomposition modulo $\mathscr{H}$* if the left (right) cosets form a partition (see [15, Definition I.2.10]). If $\mathscr{L}$ has left and right coset decompositions modulo $\mathscr{H}$, then we say that $\mathscr{L}$ has a coset decomposition modulo $\mathscr{H}$.

The proof of the next proposition can be found in Theorem I.2.12 of Pflugfelder [15].

Proposition 3.4. *A loop $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}$ if and only if for any $x \in \mathscr{L}$ and $h \in \mathscr{H}$, $(xh)\mathscr{H} = x\mathscr{H}$.*

Remark 3.5. By the definition of normal subloop, a loop has coset decomposition modulo its normal subloops. Similarly, a loop has coset decomposition modulo its nucleus. This can be seen as follows. Given $x \in \mathscr{L}$ and $n, n_1 \in \mathrm{Nuc}(\mathscr{L})$, then

$$(xn)n_1 = x(nn_1), \text{ thus } (xn)\mathrm{Nuc}(\mathscr{L}) = x\mathrm{Nuc}(\mathscr{L}),$$

and similarly for the right-hand side.

If a loop has a left coset decomposition modulo a subloop, then we can define a left index of the subloop in the loop.

Definition 3.6. Let $\mathscr{L}$ be a loop and $\mathscr{H}$ a subloop with $\mathscr{L}$ having a left coset decomposition modulo $\mathscr{H}$, then a *left transversal* X of $\mathscr{H}$ is a set of representatives, one from each left coset. The *left index*, n, of $\mathscr{H}$ in $\mathscr{L}$ is the cardinality of X, denoted by $[\mathscr{L} : \mathscr{H}]_l = n$, where n is finite or infinite (note this is well defined since the left cosets form a partition).

Lemma 3.7. *Let $\mathscr{L}$ be a loop and $\mathscr{H}, \mathscr{K}$ subloops of $\mathscr{L}$. If $\mathscr{L}$ has left coset decompositions modulo $\mathscr{H}$ and $\mathscr{K}$, then $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H} \cap \mathscr{K}$ and if $x\mathscr{H} \cap y\mathscr{K}$ for $x, y \in \mathscr{L}$ is non-empty, then $x\mathscr{H} \cap y\mathscr{K}$ is a left coset of $\mathscr{H} \cap \mathscr{K}$ in $\mathscr{L}$.*

Proof. Let $x \in \mathscr{L}$ and $t \in \mathscr{H} \cap \mathscr{K}$. By Lemma 3.2 and Proposition 3.4 we have

$$x(\mathscr{H} \cap \mathscr{K}) = x\mathscr{H} \cap x\mathscr{K} = (xt)\mathscr{H} \cap (xt)\mathscr{K} = (xt)(\mathscr{H} \cap \mathscr{K}).$$

By Proposition 3.4 it follows that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H} \cap \mathscr{K}$. This proves the first part of our claim.

Assume now that $x\mathscr{H} \cap y\mathscr{K}$ is non-empty and $z \in x\mathscr{H} \cap y\mathscr{K}$. By Proposition 3.4 it follows $z\mathscr{H} = x\mathscr{H}$ and $z\mathscr{K} = y\mathscr{K}$. Thus $z\mathscr{H} \cap z\mathscr{K} = x\mathscr{H} \cap y\mathscr{K}$. By Lemma 3.2 we obtain $x\mathscr{H} \cap y\mathscr{K} = z(\mathscr{H} \cap \mathscr{K})$. □

Proposition 3.8. *Let $\mathscr{L}$ be a loop and $\mathscr{H}$ and $\mathscr{K}$ subloops of $\mathscr{L}$ with $\mathscr{L}$ having left coset decompositions modulo $\mathscr{H}$ and $\mathscr{K}$. If $\mathscr{H}$ and $\mathscr{K}$ have finite left index in $\mathscr{L}$, then $\mathscr{H} \cap \mathscr{K}$ has finite left index in $\mathscr{L}$. Specifically*

$$[\mathscr{L} : \mathscr{H} \cap \mathscr{K}]_l \leqslant [\mathscr{L} : \mathscr{H}]_l[\mathscr{L} : \mathscr{K}]_l.$$

Proof. To each coset $x(\mathscr{H} \cap \mathscr{K})$ assign the ordered pair of cosets $(x\mathscr{H}, x\mathscr{K})$. Since by Proposition 3.4 we have $(xh)\mathscr{H} = x\mathscr{H}$ and $(xk)\mathscr{K} = x\mathscr{K}$ for all $h \in \mathscr{H}$, $k \in \mathscr{K}$, this assignment is well defined. We have to show that the mapping $x(\mathscr{H} \cap \mathscr{K}) \to (x\mathscr{H}, x\mathscr{K})$, is an injection. Let $(x\mathscr{H}, x\mathscr{K}) = (y\mathscr{H}, y\mathscr{K})$ and suppose $x = yh = yk$ for some $h \in \mathscr{H}, k \in \mathscr{K}$. By cancellation we conclude $h = k$. By Proposition 3.4 it follows that $x(\mathscr{H} \cap \mathscr{K}) = y(\mathscr{H} \cap \mathscr{K})$. □

The following corollary is Poincaré's Theorem for loops.

Corollary 3.9. *Let $\mathscr{L}$ be a loop and $\mathscr{H}_1, \ldots, \mathscr{H}_n$ subloops of $\mathscr{L}$ with $\mathscr{L}$ having left coset decompositions modulo $\mathscr{H}_1, \ldots, \mathscr{H}_n$. If $\mathscr{H}_1, \ldots, \mathscr{H}_n$ each have finite left index in $\mathscr{L}$, then $\mathscr{H}_1 \cap \ldots \cap \mathscr{H}_n$ has finite left index in $\mathscr{L}$.*

Given a loop $\mathscr{L}$ and a subloop $\mathscr{H}$, then it is clear that $\mathscr{L} = \bigcup_{x \in \mathscr{L}} x\mathscr{H}$ so $\mathscr{L}$ is the (not necessarily disjoint) union of cosets of $\mathscr{H}$. If $\mathscr{L}$ is a union of a family $\mathscr{F}$ of cosets of $\mathscr{H}$, we say that $\mathscr{F}$ is *irredundant*, if $A \in \mathscr{F}$ implies that A is not contained in any other $B \in \mathscr{F}$. We observe that this is a covering of $\mathscr{L}$ by *minimal complexes* of $\mathscr{L}$ in the sense of Steinberger [18]. This leads to a weaker notion of index of which we will make use in the example of the next section.

Definition 3.10. If $\mathscr{L}$ is a union of a finite family $\mathscr{F}$ of irredundant cosets of $\mathscr{H}$, then the *left covering index* of $\mathscr{H}$ in $\mathscr{L}$ is defined as

$$[\mathscr{L}:\mathscr{H}]^*_{\mathrm{l}} = \min\{|\mathscr{F}| : \mathscr{F} \text{ a finite irredundant covering of } \mathscr{L} \text{ by cosets of } \mathscr{H}\}.$$

If $\mathscr{L}$ has no covering by a finite family $\mathscr{F}$ of irredundant cosets of $\mathscr{H}$, then we say $[\mathscr{L}:\mathscr{H}]^*_{\mathrm{l}}$ is infinite. Similarly, we define the *right covering index* and denote it by $[\mathscr{L}:\mathscr{H}]^*_{\mathrm{r}}$.

4. An example

In this section, we provide an example of an infinite non-commutative loop which is the union of three proper abelian subgroups. This loop has no finite homomorphic image, in particular none which is isomorphic to the Klein 4-group, and its center is trivial. However, each of the abelian subgroups in the covering has covering index 3 in the loop.

Example 4.1. Consider a field $\mathbb{F}$ with multiplicative group $\mathbb{F}^*$ and the idempotent quasigroup with binary operation $\odot$ given in the table below:

$\odot$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Let $\mathscr{L}^{(3)}(\mathbb{F}) = \{a_i(x) : x \in \mathbb{F}^* \text{ and } i = 1, 2, 3\} \cup \{\mathbf{1}\}$ (i.e. each element of the form $a_i(x)$ in this set is double indexed by i and x). We define a binary operation on $\mathscr{L}^{(3)}(\mathbb{F})$ as follows:

(i) For any $l \in \mathscr{L}^{(3)}(\mathbb{F})$, $\mathbf{1}l = l\mathbf{1} = l$;

(ii) For $x, y \in \mathbb{F}^*$,

$$a_i(x)a_i(y) = \begin{cases} a_i(x+y) & \text{if } x+y \neq 0, \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

(iii) For $x, y \in \mathbb{F}^*$, $a_i(x)a_j(y) = a_{i\odot j}(xy)$ for $i < j$, and $a_i(x)a_j(y) = a_{i\odot j}(-xy)$ for $i > j$.

Then $\mathscr{L}^{(3)}(\mathbb{F})$ is a loop.

Proof. By definition, the element $\mathbf{1}$ is a two-sided identity. For convenience in light of ii., we will also denote $\mathbf{1}$ by $a_i(0)$, where $i \in \{1, 2, 3\}$, and thus if $x + (-x) = 0$ we get $a_i(x)a_i(-x) = a_i(0) = \mathbf{1}$. It remains to be shown that $a_i(x)b = a_j(y)$ has a unique solution b for all $a_i(x), a_j(y) \in \mathscr{L}^{(3)}(\mathbb{F})$. If $i = j$, then the unique solution is $b = a_i(y - x)$. Now if $i \neq j$, there exists a unique k such that $i \odot k = j$, and we can assume $x \neq 0$. Then for $i < k$, the unique solution is $b = a_k(x^{-1}y)$, and for $i > k$, the unique solution is $b = a_k(-x^{-1}y)$. Similarly, we can find a unique solution b for $ba_i(x) = a_j(y)$, thus $\mathscr{L}^{(3)}(\mathbb{F})$ is a loop. $\square$

Proposition 4.2. *Consider* $\mathscr{L}^{(3)}(\mathbb{F})$. *Then*

(4.2.1) $A_i = \{a_i(x) : x \in \mathbb{F}\}$, $i = 1, 2, 3$, *are abelian subgroups of* $\mathscr{L}^{(3)}(\mathbb{F})$;
(4.2.2) $\mathscr{L}^{(3)}(\mathbb{F}) = A_1 \cup A_2 \cup A_3$ *and* $A_i \cap A_j = \{\mathbf{1}\}$, $i \neq j$;
(4.2.3) $\mathscr{L}^{(3)}(\mathbb{F})$ *is power-associative*;
(4.2.4) $\mathscr{L}^{(3)}(\mathbb{F}) = A_i\mathbf{1} \cup A_i a_j(1) \cup A_i a_k(1)$ *where* i, j *and* k *are distinct, i.e. it is the union of three cosets of* A_i. *For* char $\mathbb{F} = 0$, *there exist infinite sets y with* $\mathscr{L}^{(3)}(\mathbb{F}) = \bigcup_{y \in Y} yA_i$, *but no proper subset of y gives a covering of* $\mathscr{L}^{(3)}(\mathbb{F})$.

Proof. Note that for ease of notation in the definition of A_i we set again $a_i(0) = \mathbf{1}$. Now (4.2.1) and (4.2.2) follow directly from the definition of $\mathscr{L}^{(3)}(\mathbb{F})$. By (4.2.1) and (4.2.2) each $l \in \mathscr{L}^{(3)}(\mathbb{F})$ is contained in a group. It follows that the subloop generated by l is a group. Hence (4.2.3) holds. The first part of (4.2.4) is obvious. If char $\mathbb{F} = 0$, consider $Y = A_j$, $j \neq i$. □

In view of (4.2.4) and Definition 3.10 we have the following corollary.

Corollary 4.3. *If* char $\mathbb{F} = 0$, *then* $[\mathscr{L}^{(3)}(\mathbb{F}) : A_i]_r^* = 3$.

Proposition 4.4. *For* $\mathscr{L}^{(3)}(\mathbb{F})$ *the following hold*:

(4.4.1) *If* $|\mathbb{F}| < \infty$, *then* $|\mathscr{L}^{(3)}(\mathbb{F})| = 3|\mathbb{F}| - 2$;
(4.4.2) $\mathscr{L}^{(3)}(\mathbb{F})$ *is a group if and only if* $|\mathbb{F}| = 2$, *and then* $\mathscr{L}^{(3)}(\mathbb{F})$ *is isomorphic to the Klein* 4-*group*;
(4.4.3) *If* char $\mathbb{F} \neq 2$, *then* $\mathscr{L}^{(3)}(\mathbb{F})$ *has a trivial centrum*.

Proof. The first claim follows from (4.2.2). If $\mathbb{F} = GF(2)$, then $|\mathscr{L}^{(3)}(\mathbb{F})| = 4$ and for any $x \in \mathscr{L}^{(3)}(\mathbb{F})$, $x^2 = \mathbf{1}$. We conclude that $\mathscr{L}^{(3)}(\mathbb{F})$ is the Klein 4-group. If $|\mathbb{F}| > 2$, then there exists $x \in \mathbb{F}^*$ with $-x^2 \neq 1$. In this case $a_1(-x)(a_1(x)a_2(1)) = a_2(-x^2) \neq a_2(1)$. It follows that $\mathscr{L}^{(3)}(\mathbb{F})$ does not have the left inverse property. Therefore, it is not associative and hence not a group.

To prove (4.4.3), consider $b \in C(\mathscr{L}^{(3)}(\mathbb{F}))$. Then $ba_j(y) = a_j(y)b$ for all $y \in \mathbb{F}$ and $j = 1, 2, 3$. Suppose $b = a_i(x)$ for some i and some $x \in \mathbb{F}$. Consider $j \neq i$ and $y = 1$. Then $a_i(x)a_j(1) = a_j(1)a_i(x)$, or $a_{i \odot j}(x) = a_{i \odot j}(-x)$. It follows that $2x = 0$, a contradiction unless $x = 0$. We conclude $b = a_i(0) = \mathbf{1}$. □

Of interest in this context is the following proposition.

Proposition 4.5. *If* char $\mathbb{F} = 0$, *then* $\mathscr{L}^{(3)}(\mathbb{F})$ *has no normal subloop of finite index*.

Proof. Let char $\mathbb{F} = 0$ and $\mathscr{K}$ be a normal subloop of $\mathscr{L}^{(3)}(\mathbb{F})$ of finite index. Set $n = |\mathscr{L}^{(3)}(\mathbb{F})/\mathscr{K}|$. Then for every $a_i(x) \in \mathscr{L}^{(3)}(\mathbb{F})$, there exists a positive integer $k \leqslant n$ such that $(a_i(x))^k \in \mathscr{K}$, since $\mathscr{L}^{(3)}(\mathbb{F})/\mathscr{K}$ is a power associative loop, and thus the order of any element in $\mathscr{L}^{(3)}(\mathbb{F})/\mathscr{K} \leqslant n$. It follows $l^{n!} \in \mathscr{K}$ for all $l \in \mathscr{L}^{(3)}(\mathbb{F})$. Since char $\mathbb{F}$ is zero, $x \in \mathbb{F}$ implies $\frac{x}{n!} \in \mathbb{F}$. Let $l = a_i(\frac{x}{n!})$. Then $l^{n!} = (a_i(\frac{x}{n!}))^{n!} = a_i(\frac{n!x}{n!}) = a_i(x) \in \mathscr{K}$, thus $\mathscr{K} = \mathscr{L}^{(3)}(\mathbb{F})$. □

We want to conclude this section with two remarks. If $|\mathbb{F}| = p$, where p is an odd prime, then $\mathscr{L}^{(3)}(\mathbb{F})$ is a simple loop (see [8, Theorem 5.12]). Furthermore, if we replace the idempotent quasigroup of order 3 in Example 4.1 by an idempotent quasigroup of order n for any $n > 3$, we will get a power associative loop that is a union of n distinct abelian subgroups with trivial intersections [8].

5. *n*-Coverings and finite coverings

Following [3], we make the following definition.

Definition 5.1. A loop $\mathscr{L}$ has an n-covering, if there exist subloops $\mathscr{H}_i$, $i \in \Omega$, an index set, such that for every $\{x_1, \ldots, x_n\} \subseteq \mathscr{L}$ there exists an $i \in \Omega$ with $\{x_1, \ldots, x_n\} \subseteq \mathscr{H}_i$. A 1-covering of a loop is called a covering, if all subloops are proper. An n-covering is finite if Ω is finite.

In [3], it was shown that a group has a finite n-covering by subgroups if and only if it has a finite homomorphic image whose minimal number of generators is greater than n. In our context n-coverings by subgroups are of interest, since they lead to certain associativity conditions. This is the content of the next theorem.

Theorem 5.2. *Let $\mathscr{L}$ be a loop, then*

(i) *$\mathscr{L}$ has a 1-covering by subgroups if and only if it is power-associative*;
(ii) *$\mathscr{L}$ has a 2-covering by subgroups if and only if it is diassociative*;
(iii) *$\mathscr{L}$ has a 3-covering by subgroups if and only if it is a group*.

Proof. To show (i), we observe that if $\mathscr{L}$ has a 1-covering by subgroups, then $\mathscr{L}$ has a covering by its cyclic subgroups, hence $\mathscr{L}$ is power-associative. Conversely, if $\mathscr{L}$ is power-associative, it has a 1-covering by its cyclic subgroups.

Now, let $\mathscr{L}$ have a 2-covering by subgroups. Then, given $a, b \in \mathscr{L}$, there exists a subgroup $\mathscr{H}$ of $\mathscr{L}$ with $a, b \in \mathscr{H}$, hence $\langle a, b\rangle$, the smallest subloop containing a and b, is a subgroup and $\mathscr{L}$ is diassociative. Conversely, if $\mathscr{L}$ is diassociative, then $\langle a, b\rangle$ is a group for all $a, b \in \mathscr{L}$. Hence $\mathscr{L}$ has a 2-covering. Thus (ii) holds.

Finally, let $\mathscr{L}$ have a 3-covering by subgroups, then, given $a, b, c \in \mathscr{L}$, there exists a subgroup $\mathscr{H}$ of $\mathscr{L}$ with $a, b, c \in \mathscr{H}$. Hence $\langle a, b, c\rangle$ is a group so $\mathscr{L}$ is associative. Conversely, if $\mathscr{L}$ is a group, then obviously $\mathscr{L}$ has a 3-covering by subgroups of $\mathscr{L}$. We conclude that (iii) holds. □

We turn now to the discussion of finite coverings and finite n-coverings of loops. Recall that a loop has a finite covering if it is the union of finitely many proper subloops, and similarly, a loop has a finite *n*-covering if the finite covering is an *n*-covering. First, we show that the analogue of the result that a group never is the union of two proper subgroups carries over directly not only to loops but even to quasigroups.

Theorem 5.3. *A quasigroup is never the union of two proper subquasigroups.*

Proof. Suppose that $Q = A \cup B$, where Q is a quasigroup and A and B are proper subquasigroups. If $X = A - (A \cap B)$ and $Y = B - (A \cap B)$, then X and Y are non-empty. Let $a \in X$ and $b \in Y$, then $ab \in Q$. Without loss of generality we may assume that $ab \in A$, i.e. $ab = a' \in A$. Since A is a quasigroup, there exists a unique $x \in A$ such that $ax = a'$. By cancellation, $b = x$, hence $b \in A$, a contradiction. $\square$

In our next proposition we will make use of the following definition.

Definition 5.4. An element a of a loop $\mathscr{L}$ is called *diassociative* if for any $x \in \mathscr{L}$ it follows that $\langle x, a \rangle$ is a group.

Proposition 5.5. *Given a loop $\mathscr{L}$ with a finite covering by subgroups $\mathscr{H}_i$, $i = 1, \ldots, n$, of finite left index such that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}_i$ for all i, then $\mathscr{L}$ is a power-associative loop with a subgroup $\mathscr{H}$ of finite left index in $\mathscr{L}$ and every element of $\mathscr{H}$ is diassociative.*

Proof. By (i) of Theorem 5.2, $\mathscr{L}$ is power-associative. Let $\mathscr{H} = \mathscr{H}_1 \cap \ldots \cap \mathscr{H}_n$. Corollary 3.9 implies that $\mathscr{H}$ has finite left index in $\mathscr{L}$. Let $a \in \mathscr{H}$ and $x \in \mathscr{L}$. Then there exists an i such that $x \in \mathscr{H}_i$. Since $a \in \mathscr{H}_i$, it follows that $\langle x, a \rangle$ is a group. $\square$

Our next example shows that a loop satisfying the assumptions of Proposition 5.5 is not necessarily a group.

Example 5.6. Let $\mathscr{L} = \mathscr{H} \times G$, where $\mathscr{H}$ is a finite non-associative power-alternative loop and G is a group, then $\mathscr{L} = \bigcup_{x \in \mathscr{H}} \mathscr{H}_x$, where $\mathscr{H}_x = \langle x \rangle \times G$, is a finite covering of $\mathscr{L}$, and $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}_x$ for each x. Furthermore, $\mathscr{L}/\mathscr{H}$ has left coset decomposition modulo any cyclic subgroup, since it is a power-alternative loop.

Proposition 5.7. *Given a loop $\mathscr{L}$ with a finite* 2*-covering by subgroups $\mathscr{H}_i$, $i = 1, \ldots, n$, of finite index such that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}_i$ for all i, then $\mathscr{L}$ is a diassociative loop, and* $\mathrm{Nuc}(\mathscr{L})$ *is a subgroup of finite index in $\mathscr{L}$.*

Proof. By (ii) of Theorem 5.2, $\mathscr{L}$ is diassociative. Let $\mathscr{H} = \mathscr{H}_1 \cap \ldots \cap \mathscr{H}_n$. Corollary 3.9 implies that $\mathscr{H}$ has finite left index in $\mathscr{L}$. Let $a \in \mathscr{H}$ and $x, y \in \mathscr{L}$. Then there exists an i such that $x, y \in \mathscr{H}_i$. Since $a \in \mathscr{H}_i$, it follows that $\langle x, y, a \rangle$ is a group. Thus $a \in \mathrm{Nuc}(\mathscr{L})$, i.e. $\mathscr{H} \subseteq \mathrm{Nuc}(\mathscr{L})$. It follows that $\mathrm{Nuc}(\mathscr{L})$ has finite left index in $\mathscr{L}$, and thus $\mathrm{Nuc}(\mathscr{L})$ has finite index in $\mathscr{L}$. $\square$

Similarly as before, we provide here an example of a loop satisfying the assumptions of Proposition 5.7 which is not a group.

Example 5.8. Let $\mathscr{L} = \mathscr{H} \times G$, where $\mathscr{H}$ is a finite non-associative Moufang loop with every element of odd order and G is a group, then $\mathscr{L} = \bigcup_{\{x,y\}\subseteq\mathscr{H}} \mathscr{H}_{\{x,y\}}$, where $\mathscr{H}_{\{x,y\}} = \langle x, y\rangle \times G$, is a finite 2-covering of $\mathscr{L}$, and $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}_{\{x,y\}}$ for all $\{x, y\}$.

In the case that we have a finite 2-covering by abelian subgroups we are guaranteed a finite homomorphic image as the next corollary tells us.

Corollary 5.9. *Given a loop $\mathscr{L}$ with a finite* 2*-covering by abelian subgroups $\mathscr{H}_i$, $i = 1, \dots, n$, of finite left index such that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}_i$ for all i, then $Z(\mathscr{L})$ is of finite index in $\mathscr{L}$ as a normal subgroup of $\mathscr{L}$.*

Proof. Let $\mathscr{H} = \mathscr{H}_1 \cap \dots \cap \mathscr{H}_n$. By Proposition 5.7 we have that $[\mathscr{L} : \mathscr{H}]_l$ and $[\mathscr{L} : \mathrm{Nuc}(\mathscr{L})]$ are finite. For $x \in \mathscr{L}$ there exists an i such that $x \in \mathscr{H}_i$. Since $\mathscr{H} \subseteq \mathscr{H}_i$ and $\mathscr{H}_i$ is an abelian subgroup, we have $ax = xa$ for all $a \in \mathscr{H}$ and all $x \in \mathscr{L}$. Hence $\mathscr{H} \subseteq \mathrm{Nuc}(\mathscr{L}) \cap C(\mathscr{L}) = Z(\mathscr{L})$. It follows that $Z(\mathscr{L})$ has finite index in $\mathscr{L}$. □

We mention here that choosing the loop $\mathscr{H}$ as in Example 5.8, but in addition commutative and G as an abelian group, provides us with a loop which is not a group and satisfies the assumptions of the preceding corollary. As can be seen from the next proposition, a normal nucleus of finite index in a power-alternative loop guarantees the existence of a finite covering by subgroups.

Proposition 5.10. *If $\mathscr{L}$ is a power-alternative loop with* $\mathrm{Nuc}(\mathscr{L})$ *a normal subgroup of finite index in $\mathscr{L}$ and $\mathscr{L}/\mathrm{Nuc}(\mathscr{L})$ not cyclic, then $\mathscr{L}$ has a finite covering by subgroups $\mathscr{H}_i$ of finite index such that $\mathscr{L}$ has a coset decomposition modulo $\mathscr{H}_i$ for all i.*

Proof. Since $\mathrm{Nuc}(\mathscr{L})$ is a normal subgroup of $\mathscr{L}$, the quotient $\mathscr{L}/\mathrm{Nuc}(\mathscr{L})$ is defined. Set $\mathrm{Nuc}(\mathscr{L}) = N$ and let $\mathscr{H} = \langle g, N\rangle$ for some $g \in \mathscr{L}$. We observe that $\mathscr{H}$ is a subgroup of $\mathscr{L}$ and any $h \in \mathscr{H}$ can be written as $h = g^j n$, j an integer and $n \in N$. We claim that $\mathscr{L}$ has coset decomposition modulo $\mathscr{H}$. By Proposition 3.4 it suffices to show $(xh)\mathscr{H} = x\mathscr{H}$ and $\mathscr{H}(hx) = \mathscr{H}x$ for all $h \in \mathscr{H}$ and all $x \in \mathscr{L}$. Let $h, h_1 \in \mathscr{H}$ with $h = g^j n$, $n \in N$, then $(xh)h_1 = (x(g^j n))h_1 = ((xg^j)n)h_1 = (xg^j)(nh_1)$. Since $nh_1 \in \mathscr{H}$, we have $nh_1 = g^i n'$ for some $n' \in N$. Hence $(xg^j)(nh_1) = (xg^j)(g^i n') = (xg^{i+j})n' = x(g^{i+j}n') = xh_2$, where $h_2 \in \mathscr{H}$. It follows that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}$. The proof for right coset decomposition is similar.

Let $X = \{x_1, \dots, x_n\}$ be a left transversal of $\mathrm{Nuc}(\mathscr{L})$. Consider $\mathscr{H}_i = \langle x_i, \mathrm{Nuc}(\mathscr{L})\rangle$. Then, by the above, $\mathscr{H}_i$ is a subgroup of $\mathscr{L}$ and $\mathscr{L}$ has a coset decomposition modulo $\mathscr{H}_i$. Obviously, $\mathscr{L} = \bigcup_{i=1}^{n} \mathscr{H}_i$ so $\mathscr{L}$ has a covering. Since $\mathscr{L}/\mathrm{Nuc}(\mathscr{L})$ is not cyclic and $[\mathscr{L} : \mathscr{H}_i] < [\mathscr{L} : \mathrm{Nuc}(\mathscr{L})]$, each $\mathscr{H}_i$ is a proper subgroup of $\mathscr{L}$ of finite index. □

Since every diassociative loop is a power-alternative loop, the following corollary is a partial converse of Proposition 5.7 and Corollary 5.9.

Corollary 5.11. *If $\mathscr{L}$ is a diassociative loop with* $\mathrm{Nuc}(\mathscr{L})$ *a normal subgroup of finite index in $\mathscr{L}$ and $\mathscr{L}/\mathrm{Nuc}(\mathscr{L})$ is not cyclic, then $\mathscr{L}$ has a finite covering by subgroups $\mathscr{H}_i$*

of finite index such that $\mathscr{L}$ has a coset decomposition modulo $\mathscr{H}_i$ for all i. Furthermore, if $Z(\mathscr{L})$ has finite index in $\mathscr{L}$, then $\mathscr{L}$ is the union of finitely many abelian subgroups, each having finite index in $\mathscr{L}$.

6. Neumann's Lemma

In this section of the paper we will prove a loop analogue of Neumann's lemma. Towards that end we need to strengthen our conditions on coset decompositions modulo a subloop as given in the following definition.

Definition 6.1. A loop $\mathscr{L}$ has a *strong left* (*right*) *coset decomposition modulo* $\mathscr{H}$, where $\mathscr{H}$ is subloop of $\mathscr{L}$, *if* $y(a\mathscr{H}) = (ya)\mathscr{H}$ for all $y, a \in \mathscr{L}$. If $\mathscr{L}$ has strong left and right coset decompositions modulo $\mathscr{H}$, then we say that $\mathscr{L}$ has a strong coset decomposition modulo $\mathscr{H}$.

Lemma 6.2. *Let $\mathscr{L}$ be a loop and $\mathscr{H}$ a subloop of $\mathscr{L}$. Then $\mathscr{L}$ has a strong left coset decomposition modulo $\mathscr{H}$ if and only if it has a left coset decomposition modulo $\mathscr{H}$ and given that $\{a_i\mathscr{H}\}$ is a coset decomposition of $\mathscr{L}$ modulo $\mathscr{H}$, then so is $\{y(a_i\mathscr{H})\}$ for any $y \in \mathscr{L}$.*

Proof. Assume $\mathscr{L}$ has a strong left coset decomposition modulo $\mathscr{H}$. By Proposition 3.4 and Definition 6.1, it is obvious that $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}$. Since $y(a_i\mathscr{H}) = (ya_i)\mathscr{H}$, we observe that $y(a_i\mathscr{H})$ is a coset of $\mathscr{H}$. Assume $y(a_i\mathscr{H}) = y(a_j\mathscr{H})$, then $ya_i = y(a_jh)$ so $a_i = a_jh$, and $i = j$. Thus if $\mathscr{L} = \bigcup_{i\in I} a_i H$, then $\mathscr{L} = y\mathscr{L} = \bigcup_{i\in I} ya_i H$.

Now let $\mathscr{L}$ have a left coset decomposition modulo $\mathscr{H}$ and assume that if $\{a_i\mathscr{H}\}$ is a coset decomposition of $\mathscr{L}$ modulo $\mathscr{H}$, then so is $\{y(a_i\mathscr{H})\}$ for any $y \in \mathscr{L}$. Then for any $y, a \in \mathscr{L}$, $y(a\mathscr{H})$ is a left coset of $\mathscr{H}$ and so is $(ya)\mathscr{H}$, but $ya \in y(a\mathscr{H}) \cap (ya)\mathscr{H}$. Therefore, since $\mathscr{L}$ has a left coset decomposition modulo $\mathscr{H}$, $y(a\mathscr{H}) = (ya)\mathscr{H}$ and $\mathscr{L}$ has a strong left coset decomposition modulo $\mathscr{H}$. □

As is clear from the proof of Lemma 6.2, $\{y(a_i\mathscr{H})\}$ is a partition of $\mathscr{L}$ if $\{a_i\mathscr{H}\}$ is a coset decomposition of $\mathscr{L}$ modulo $\mathscr{H}$. However $\{y(a_i\mathscr{H})\}$ is not necessarily a coset decomposition.

Lemma 6.3. *Let $\mathscr{L}$ be a loop and $\mathscr{H}$, $\mathscr{K}$ subloops of $\mathscr{L}$ with $\mathscr{K} \leqslant \mathscr{H}$, and let $\mathscr{L}$ have left coset decompositions modulo $\mathscr{H}$ and $\mathscr{K}$. If $[\mathscr{L} : \mathscr{K}]_l$ is finite, then $\mathscr{H}$ has a left coset decomposition modulo $\mathscr{K}$ and $[\mathscr{H} : \mathscr{K}]_l$ is finite.*

Proof. By assumption we have $\mathscr{L} = \bigcup_{i=1}^{n}{}^{*} a_i\mathscr{K}$, where $\bigcup^{*}$ denotes a disjoint union. Intersecting with $\mathscr{H}$ and distributing lead to

$$\mathscr{H} = \mathscr{H} \cap \mathscr{L} = \left(\bigcup_{i=1}^{n}{}^{*} a_i\mathscr{K}\right) \cap \mathscr{H} = \bigcup_{i=1}^{n}{}^{*} (a_i\mathscr{K} \cap \mathscr{H}).$$

We claim now that $a_i\mathscr{K} \cap \mathscr{H}$ is non-empty if and only if $a_i \in \mathscr{H}$. Suppose $a_i \in \mathscr{H}$, then $a_i\mathscr{K} \subseteq \mathscr{H}$ and $a_i\mathscr{K} \cap \mathscr{H} = a_i\mathscr{K}$. Conversely, suppose $a_i\mathscr{K} \cap \mathscr{H}$ is non-empty. Then there exist $k \in \mathscr{K}$ and $h \in \mathscr{H}$ such that $a_i k = h$. Since $\mathscr{H}$ is a subloop of $\mathscr{L}$, there exists a unique $x \in \mathscr{H}$ such that $xk = h$. By cancellation we obtain $x = a_i$, hence $a_i \in \mathscr{H}$. We relabel now the transversal $\{a_1, \ldots, a_m, a_{m+1}, \ldots, a_n\}$ such that $a_i\mathscr{K} \cap \mathscr{H}$ is non-empty for $i \leqslant m$ and empty for $i > m$. By Lemma 3.7 and the above it then follows

$$\mathscr{H} = \bigcup_{i=1}^{m\ *} (a_i\mathscr{K} \cap \mathscr{H}) = \bigcup_{i=1}^{m\ *} a_i\mathscr{K}$$

with $a_i \in \mathscr{H}$. We conclude that $\mathscr{H}$ has left coset decomposition modulo $\mathscr{K}$ and $[\mathscr{H} : \mathscr{K}]_l$ is finite.

Before stating our main result, we note that Lemma 3.7 and 6.3 hold as well for a strong left coset decomposition.

Theorem 6.4. *Let $\mathscr{L}$ be a loop with $\mathscr{L} = \bigcup_{i=1}^{n} g_i\mathscr{H}_i$, where $\mathscr{H}_1, \ldots, \mathscr{H}_n$ are* (*not necessarily distinct*) *subloops of $\mathscr{L}$ and with $\mathscr{L}$ having strong left coset decompositions modulo $\mathscr{H}_i, i = 1, \ldots, n$. Then all cosets in this union for which the corresponding index $[\mathscr{L} : \mathscr{H}_i]_l$ is infinite can be omitted from the union and the remaining cosets still cover the loop.*

Proof. We first show that at least one $\mathscr{H}_i$ has finite left index in $\mathscr{L}$. Let $\mathscr{L} = \bigcup_{i=1}^{n} g_i\mathscr{H}_i$ and assume that r of the $\mathscr{H}_1, \ldots, \mathscr{H}_n$ are distinct. We prove our claim by induction on r. If $r = 1$, then $\mathscr{L}$ is a union of finitely many left cosets of the subloop $\mathscr{H}_1$ and $[\mathscr{L} : \mathscr{H}_1]_l$ is finite. Now let $r > 1$ and assume if $r - 1$ subloops are distinct, then at least one has finite left index in $\mathscr{L}$.

Assume that the $\mathscr{H}_i$ are labeled such that $\mathscr{H}_{m+1} = \cdots = \mathscr{H}_n$ with $m < n$ and $\mathscr{H}_n$ is distinct from each of $\mathscr{H}_1, \ldots, \mathscr{H}_m$, and where exactly $r - 1$ of the first m subloops are distinct, and so $r - 1 \leqslant m$. Therefore we have

$$\mathscr{L} = \left(\bigcup_{i=1}^{m} g_i\mathscr{H}_i\right) \cup \left(\bigcup_{i=m+1}^{n} g_i\mathscr{H}_n\right).$$

If $\mathscr{L} = \bigcup_{i=m+1}^{n} g_i\mathscr{H}_n$, then $[\mathscr{L} : \mathscr{H}_n]_l$ is finite and the subloops $\mathscr{H}_1, \ldots, \mathscr{H}_m$ can be omitted. If not, then there exists $x \in \mathscr{L}$ such that

$$x \in \bigcup_{i=1}^{m} g_i\mathscr{H}_i, \quad \text{but } x \notin \bigcup_{i=m+1}^{n} g_i\mathscr{H}_i.$$

We claim

$$x\mathscr{H}_n \subseteq \bigcup_{i=1}^{m} g_i\mathscr{H}_i. \tag{6.1}$$

Suppose to the contrary. Then there exists $h \in \mathscr{H}_n$ with $xh \notin \bigcup_{i=1}^{m} g_i\mathscr{H}_i$. Thus $xh \in \bigcup_{i=m+1}^{n} g_i\mathscr{H}_n$ and so, using Proposition 3.4, we obtain $x\mathscr{H}_n = g_j\mathscr{H}_n$ for some j with $m + 1 \leqslant j \leqslant n$. We conclude that $x \in \bigcup_{i=m+1}^{n} g_i\mathscr{H}_i$, a contradiction. Thus (6.1) holds.

Let u_j be the unique solution of $u_j x = g_j$, so $u_j(x\mathscr{H}_n) = (u_j x)\mathscr{H}_n = g_j\mathscr{H}_n$. Left multiplication of (6.1) by u_j leads to

$$g_j\mathscr{H}_n \subseteq u_j\left(\bigcup_{i=1}^{m} g_i\mathscr{H}_i\right) = \bigcup_{i=1}^{m}(u_j g_i)\mathscr{H}_i = \bigcup_{i=1}^{m} c_{ij}\mathscr{H}_i,$$

where $c_{ij} = u_j g_i$. We conclude that

$$\bigcup_{j=m+1}^{n} g_j\mathscr{H}_n \subseteq \bigcup_{j=m+1}^{n}\ \bigcup_{i=1}^{m} c_{ij}\mathscr{H}_i.$$

Therefore

$$\mathscr{L} = \left(\bigcup_{i=1}^{m} g_i\mathscr{H}_i\right) \cup \left(\bigcup_{j=m+1}^{n}\ \bigcup_{i=1}^{m} c_{ij}\mathscr{H}_i\right).$$

So $\mathscr{L}$ is a union of finitely many cosets of $\mathscr{H}_1, \ldots, \mathscr{H}_m$ of which now only $r-1$ are distinct by our assumption, and hence, by induction on r, at least one of the $\mathscr{H}_i$ has finite left index in $\mathscr{L}$.

Consider $\mathscr{L} = \bigcup_{i=1}^{n} g_i\mathscr{H}_i$ and assume that $\mathscr{H}_1, \ldots, \mathscr{H}_m$ have infinite left index in $\mathscr{L}$ and $\mathscr{H}_{m+1}, \ldots, \mathscr{H}_n$ have finite left index in $\mathscr{L}$. By the above, we know that $m < n$ and obtain

$$\mathscr{L} = \left(\bigcup_{i=1}^{m} g_i\mathscr{H}_i\right) \cup \left(\bigcup_{j=m+1}^{n} g_j\mathscr{H}_j\right).$$

Let $I = \mathscr{H}_{m+1} \cap \ldots \cap \mathscr{H}_n$. Since $[\mathscr{L} : \mathscr{H}_j]_l < \infty$ for $m+1 \leqslant j \leqslant n$, it follows by Corollary 3.9 that $[\mathscr{L} : I]_l$ is finite. Since $I \subseteq \mathscr{H}_j$ for $m+1 \leqslant j \leqslant n$, Lemma 6.3 implies that $[\mathscr{H}_j : I]_l = n_j < \infty$ and that $\mathscr{H}_j$ has a strong left coset decomposition modulo I. Choose a set of coset representatives of $\mathscr{H}_j$ modulo I, say $\{a_{jk}\}$, $a_{jk} \in \mathscr{H}_j$, $1 \leqslant k \leqslant n_j$. Setting $b_{jk} = g_j a_{jk}$, we see that

$$g_j\mathscr{H}_j = \bigcup_{k=1}^{n_j} g_j(a_{jk}I) = \bigcup_{k=1}^{n_j}(g_j a_{jk})I = \bigcup_{k=1}^{n_j} b_{jk}I.$$

We conclude

$$\mathscr{L} = \left(\bigcup_{i=1}^{m} g_i\mathscr{H}_i\right) \cup \left(\bigcup_{j=m+1}^{n}\ \bigcup_{k=1}^{n_j} b_{jk}I\right). \tag{6.2}$$

If

$$\mathscr{L} = \bigcup_{j=m+1}^{n} \bigcup_{k=1}^{n_j} b_{jk} I, \text{ then } \mathscr{L} = \bigcup_{j=m+1}^{n} g_j \mathscr{H}_j$$

and all the subloops of infinite left index have been omitted. Else, there exists $x \in \mathscr{L}$ such that

$$x \in \bigcup_{i=1}^{m} g_i \mathscr{H}_i \quad \text{and} \quad x \notin \bigcup_{j=m+1}^{n} \bigcup_{k=1}^{n_j} b_{jk} I.$$

We claim

$$xI \subseteq \bigcup_{i=1}^{m} g_i \mathscr{H}_i. \tag{6.3}$$

Assume to the contrary. Then there exists $h \in I$ with $xh \notin \bigcup_{i=1}^{m} g_i \mathscr{H}_i$. Therefore

$$xh \in \bigcup_{j=m+1}^{n} g_j \mathscr{H}_j.$$

Hence $xh \in g_{j'} \mathscr{H}_{j'}$ for some j', $m+1 \leqslant j' \leqslant n$. Since $I \subseteq \mathscr{H}_{j'}$, it follows that

$$x\mathscr{H}_{j'} = g_{j'} \mathscr{H}_{j'} \quad \text{and} \quad x \in \bigcup_{j=m+1}^{n} g_j \mathscr{H}_j,$$

a contradiction. This proves (6.3).

Let w_{jk} be the unique solution of $w_{jk} x = b_{jk}$, and set $d_{ijk} = w_{jk} g_i$. Left multiplication of (6.1) by w_{jk} leads to

$$w_{jk}(xI) = (w_{jk}x)I = b_{jk} I \subseteq \bigcup_{i=1}^{m} w_{jk}(g_i \mathscr{H}_i) = \bigcup_{i=1}^{m} (w_{jk} g_i) \mathscr{H}_i = \bigcup_{i=1}^{m} d_{ijk} \mathscr{H}_i.$$

Subsequently,

$$\bigcup_{j=m+1}^{n} \bigcup_{k=1}^{n_j} b_{jk} I \subseteq \bigcup_{j=m+1}^{n} \bigcup_{k=1}^{n_j} \bigcup_{i=1}^{m} d_{ijk} \mathscr{H}_i.$$

This together with (6.2) yields

$$\mathscr{L} = \left(\bigcup_{i=1}^{m} g_i \mathscr{H}_i \right) \cup \left(\bigcup_{j=m+1}^{n} \bigcup_{k=1}^{n_j} \bigcup_{i=1}^{m} d_{ijk} \mathscr{H}_i \right).$$

But this would imply that at least one of the $\mathscr{H}_1, \ldots, \mathscr{H}_m$ has finite left index in $\mathscr{L}$, a contradiction to our assumption that $\mathscr{H}_1, \ldots, \mathscr{H}_m$ all have infinite left index in $\mathscr{L}$. Thus $\mathscr{L} = \bigcup_{j=m+1}^{n} g_j \mathscr{H}_j$, the desired result. □

Now Theorem 6.4 together with Propositions 5.5, 5.7 and Corollary 5.9, respectively, leads to the following three corollaries.

Corollary 6.5. *Given a loop* $\mathscr{L}$ *with a finite covering by subgroups* $\mathscr{H}_i, i = 1, \ldots, n$, *such that* $\mathscr{L}$ *has a strong left coset decomposition modulo* $\mathscr{H}_i$ *for all i, then* $\mathscr{L}$ *is a power-associative loop with a subgroup* $\mathscr{H}$ *of finite left index in* $\mathscr{L}$ *and every element of* $\mathscr{H}$ *is diassociative*.

Corollary 6.6. *Given a loop* $\mathscr{L}$ *with a finite* 2*-covering by subgroups* $\mathscr{H}_i, i=1, \ldots, n$, *such that* $\mathscr{L}$ *has a strong left coset decomposition modulo* $\mathscr{H}_i$ *for all i, then* $\mathscr{L}$ *is a diassociative loop, and* $\mathrm{Nuc}(\mathscr{L})$ *is a subgroup of finite index in* $\mathscr{L}$.

Corollary 6.7. *Given a loop* $\mathscr{L}$ *with a finite* 2*-covering by abelian subgroups* $\mathscr{H}_i$, $i = 1, \ldots, n$, *such that* $\mathscr{L}$ *has a strong left coset decomposition modulo* $\mathscr{H}_i$ *for all i, then* $Z(\mathscr{L})$ *is of finite index in* $\mathscr{L}$.

We conclude with two examples showing that there exist loops satisfying the assumptions of Theorem 6.4, Corollaries 6.5 and 6.6 which are not necessarily groups. Since every normal subloop has strong left coset decomposition, every subloop of a Hamiltonian loop has strong left coset decomposition. Norton in [14] shows the existence of finite power associative as well as disassociative Hamiltonian loops which are not groups.

Example 6.8. Let $\mathscr{L} = \mathscr{H} \times G$ where $\mathscr{H}$ is a finite non-associative power associative Hamiltonian loop and G is a group, then $\mathscr{L} = \bigcup_{x \in \mathscr{H}} \mathscr{H}_x$, where $\mathscr{H}_x = \langle x \rangle \times G$, is a finite covering of $\mathscr{L}$, and $\mathscr{L}$ has a strong left coset decomposition modulo $\mathscr{H}_x$ for all x.

Example 6.9. Let $\mathscr{L} = \mathscr{H} \times G$ where $\mathscr{H}$ is a finite non-associative diassociative Hamiltonian loop and G is a group, then $\mathscr{L} = \bigcup_{\{x,y\} \subseteq \mathscr{H}} \mathscr{H}_{\{x,y\}}$, where $\mathscr{H}_{\{x,y\}} = \langle x, y \rangle \times G$, is a finite 2-covering of $\mathscr{L}$, and $\mathscr{L}$ has a strong left coset decomposition modulo $\mathscr{H}_{\{x,y\}}$ for all $\{x, y\}$.

In conclusion, we mention that by Norton [14] every commutative, diassociative Hamiltonian loop is an abelian group. Thus a construction similar to the ones in Examples 6.8 and 6.9 does not lead to a non-associative loop satisfying the assumptions of Corollary 6.7. Nevertheless we suspect the existence of such loops.

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