Polar Decomposition of Locally Finite Groups

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The study of loops as transversals in groups dates back to the works of Reinhold Baer. In the past few years there have been several papers using polar decomposition in linear algebra in order to construct Bruck loops. In this paper we generalize the notion of polar decomposition to any arbitrary group, and we show that in any polar decomposition the binary operation "inherited" from the group leads to the construction of a Bruck loop. We end this paper by classifying, in some sense, all locally finite groups with polar decomposition.

1. INTRODUCTION

A group G has a polar decomposition if there is an involution $\tau \in Aut(G)$, such that every element of the form $g(g^{-1})^{\tau}$ for $g \in G$ has a unique square root of the same form. If a group G has a polar decomposition due to an involution $\tau \in Aut(G)$, then G decomposes as the product of two sets, the stabilizer of τ and $P(\tau) = \{g(g^{-1})^{\tau} \colon g \in G\}$. The polar decomposition for a group is the analogue of the complex number decomposition, given $z \in \mathbb{C} - \{0\}$, then $z = re^{i\phi}$. Polar decomposition is well known in linear algebra [9]. Given a finite-dimensional complex inner product space the set of all invertible linear operators has a polar decomposition into unitary and positive operators, such that an invertible operator $T = \sqrt{TT^*}\sqrt{TT^*}(T^*)^{-1}$ where $\sqrt{TT^*}$ is a positive operator and $\sqrt{TT^*}(T^*)^{-1}$ is a unitary operator. Note that * is an anti-automorphism, but composing it with the inverse gives us an automorphism. Similarly polar decomposition is found in Functional Analysis [15], compact Lie groups [13, 14] and in the construction of Bruck loops (equivalently K-loops) [12, 10]. "Traditional" polar decomposition is based on a vector space and an inner product, this definition is based on properties of the group. Thus, the "Traditional" polar decomposition is a special case of this definition. In section 5 of [4], we look at polar decomposition for groups of odd order with an involution automorphism. In this paper we will classify, in some sense, all locally finite groups with polar decomposition.

2. POLAR DECOMPOSITION OF GROUPS

Definition 2.1. A group G has a *polar decomposition* if there exists an involution $\tau \in Aut(G)$ such that every $a \in P(\tau) = \{g(g^{-1})^{\tau} \colon g \in G\}$ has a unique square root in $P(\tau)$. We will denote by $\sqrt{a}$ the square root of $a \in P(\tau)$, and the stabilizer of τ by $C_G(\tau)$.

Definition 2.2. Given $\tau \in Aut(G)$ we define $G_\tau = G$ if $\tau \in Inn(G)$, otherwise $G_\tau = G \rtimes < \tau >$.

Remark 2. 1. Since $g(g^{-1})^{\tau} = \tau^g \tau$ in G_τ, $p(\tau)$ is the set of all products of two conjugates of τ in G.

Definition 2.3. We call a set S with a binary operation $\odot$ a *groupoid* if it is closed under $\odot$.

Definition 2.4. We call a groupoid S, a *uniquely* 2*-divisible groupoid* [5], if $a \mapsto a^2$ is a permutation of S (i.e. given an $a \in S$ there exists a unique element $\sqrt{a} \in S$ with $(\sqrt{a})^2 = a$).

Definition 2.5. A set B is a transversal in a group G (all transversals in this paper are left transversals) of a subgroup H of G if every $g \in G$ can be written uniquely as $g = bh$ where $b \in B$ and $h \in H$. Let $b_1, b_2 \in B$ be any two elements of B, and let

$$b_1 b_2 = (b_1 \odot b_2) h(b_1, b_2)$$

be the unique decomposition of the element $b_1 b_2 \in G$, where $b_1 \odot b_2 \in B$ and $h(b_1, b_2) \in H$. Equation above determines an "inherited" binary operation, $\odot$, in B, called the *left loop operation of* B induced by H, and a map $h \colon B \times B \to H$, called the *transversal map.*

A *transversal groupoid* $(B, \odot)$ of H in G is a groupoid formed by a transversal B of H in G with its loop operation $\odot$.

Definition 2.6. Given a group G and an involution $\tau \in Aut(G)$, the *inverter* of τ is $K(\tau) = \{a \in G \colon a^\tau = a^{-1}\}$.

Definition 2.7. A subset P of a group G is a *twisted subgroup* [1] of G if (i) $1 \in P$; and (ii) $aPa \subseteq P$ for all $a \in P$.

For the convenience of the reader I will restate two important theorems from [10] in the terminology of the current article. It is important to note that given a nonabelian group G any involution in $Aut(G)$ is composed with inversion is an involutory antiautomorphism [10]. Also note that the binary operations $\sqrt{xy^2x}$ and $\sqrt{x}y\sqrt{x}$ form isomorphic loops under $x \mapsto x^2$.

Theorem 2.1 (2.3 of [10]). *If G is a group and τ is an involution in $Aut(G)$, then $P(\tau)^2 \subseteq P(\tau) = P(\tau)^{-1} \subseteq K(\tau)$ and $P(\tau)$ is a twisted subgroup of G.*

Theorem 2.2. *If G is a group and τ is an involution in $Aut(G)$, then for all $p \in P(\tau)$, $< p > \subseteq P(\tau)$.*

Proof. By the proof of 1.2 [1] $p^n \in P(\tau)$ for n a positive integer. And since by 2.3 of [10], $P(\tau) = P(\tau)^{-1}$, we get that $< p > \subseteq P(\tau)$. ▮

Definition 2.8. A *loop* is a groupoid $(L, \odot)$ with an identity element in which the equations $a \odot x = b$ and $y \odot a = b$ for the unknowns x, and y possess unique solutions.

Definition 2.9. A loop B is a *Bruck loop* [6, 16] if it satisfies the following relation for all $x, y, z \in B$

1. $x \odot (y \odot (x \odot z)) = (x \odot (y \odot x)) \odot z$
2. $(x \odot y)^{-1} = x^{-1} \odot y^{-1}$

A uniquely 2-divisible Bruck loop is called a *B-loop.*

Theorem 2.3 (3.9 of [10]). *If a group G has a polar decomposition, then $P(\tau)$ under the binary operation $x \odot y = \sqrt{x}y\sqrt{x}$ is a B-loop.*

Lemma 2.1. *If a group G has an involution $\tau \in Aut(G)$ and $a \in K(\tau)$, then for $n > 0$, $a^{2n} \in P(\tau)$.*

Proof. For all $a \in K(\tau)$, $a^2 = aa = a(a^{-1})^\tau \in P(\tau)$ so by Theorem 2.2 $a^{2n} \in P(\tau)$ ▮

Lemma 2.2. *If G has a polar decomposition with respect to the involution τ in $Aut(G)$, then for all $g \in G$, $\sqrt{g(g^{-1})^\tau} g^\tau \in C_G(\tau)$.*

Proof. Let $g \in G$, then

$$(\sqrt{g(g^{-1})^\tau} g^\tau)^\tau = \sqrt{g(g^{-1})^\tau}^{-1} g = \sqrt{g(g^{-1})^\tau}\sqrt{g(g^{-1})^\tau}^{-1}\sqrt{g(g^{-1})^\tau}^{-1} g = \sqrt{g(g^{-1})^\tau}(g$$

Thus we see that $\sqrt{g(g^{-1})^\tau} g^\tau \in C_G(\tau)$. ▮

Remark 2. 2. If $g \in C_G(\tau)$, then $g = \sqrt{g(g^{-1})^\tau} g^\tau$. Thus the map $g \mapsto \sqrt{g(g^{-1})^\tau} g^\tau$ is an idempotent.

Theorem 2.4. *If a group G has a polar decomposition, then $G = P(\tau)C_G(\tau)$ and $P(\tau)$ is a transversal of $C_G(\tau)$ with left loop operation $x \odot y = \sqrt{xy^2x}$.*

Proof. Let $g \in G$ then

$$g = \sqrt{g(g^{-1})^\tau}\sqrt{g(g^{-1})^\tau} g^\tau$$

where $p = \sqrt{g(g^{-1})^\tau} \in P(\tau)$ and $c = \sqrt{g(g^{-1})^\tau} g^\tau \in C_G(\tau)$. Assume $g = p_1 c_1$ where $p_1 \in P(\tau)$, $c_1 \in C_G(\tau)$, then

$$(g^\tau)^{-1} = ((p_1c_1)^\tau)^{-1} = (p_1^\tau c_1^\tau)^{-1} = (c_1^\tau)^{-1}(p_1^\tau)^{-1} = c_1^{-1}p_1.$$

So $g(g^{-1})^\tau = p_1^2$ and by the uniqueness of the square root in $P(\tau)$,we see that p_1 and c_1 are uniquely determined. Note that if $x \in P(\tau)$, then $x^\tau = x^{-1}$. Given $x, y \in P(\tau)$, then $x \odot y = \sqrt{xy((xy)^{-1})^\tau} = \sqrt{xy(y^{-1}x^{-1})^\tau} = \sqrt{xyyx} = \sqrt{xy^2x}$. ▮

Theorem 2.5. *If a group G has a polar decomposition, then $< P(\tau) >= H$ is a normal subgroup of G.*

Proof. Since, $G = P(\tau)C_G(\tau)$ for all $g \in C_G(\tau)$ and $p \in P(\tau)$, $gpg^{-1} \in P(\tau)$. Thus, $< P(\tau) >= H$ is a normal subgroup of G. ▮

Corollary 2.1. *If a group G has a polar decomposition, then $P(\tau)$ is a B-Loop with left loop operation $x \odot y = \sqrt{xy^2x}$.*

Proof. By 3.9 of [10], and the fact that the binary operations $\sqrt{xy^2x}$ and $\sqrt{x}y\sqrt{x}$ form isomorphic loops under $x \mapsto x^2$. ▌

3. POLAR DECOMPOSITION OF LOCALLY FINITE GROUPS

Definition 3.1. Given a locally finite group G and a fixed set of primes π, then we define $O_\pi(G)$ as the maximal normal subgroup of G with all elements π-elements. And we define $O(G)$ as the maximal normal subgroup of G with all elements of odd order [11].

Lemma 3.1. *(An extension of Theorem 15 in [8] to the locally finite case) Let P be a twisted subgroup of a locally finite group G, and if all the elements of P are π-elements where π is a set of odd primes, then P generates a subgroup H of G whose elements are π-elements.*

Proof. . Let $H \leq G$ generated by P. Given $h \in H - \{1\}$, $h = p_1 \dots p_n$ where $\{p_1, \dots, p_n\} \subseteq P(\tau)$, let $K =< p_1, \dots, p_n >$, then $h \in K$ and K is a finite group generated by the twisted subgroup $P \cap K$. So, K is a π-group by Theorem 15 in [8], and h is a π-element. ▌

Theorem 3.1. *Let G be a locally finite group with a polar decomposition and if all the elements of $P(\tau)$ are π-elements where π is a set of odd primes, then $P(\tau)$ generates a subgroup H of $O_\pi(G)$.*

Proof. Lemma 3.1, and Theorem 2.5. ▌

Theorem 3.2. *Let G be a locally finite group with a polar decomposition and if all the elements of $P(\tau)$ are of odd order, then $P(\tau)$ generates a subgroup H of $O(G)$.*

Proof. Since H is a normal subgroup. ▌

Corollary 3.1. *A finite simple groups with polar decomposition is abelian.*

Lemma 3.2. *If G is a group with a polar decomposition and $p \in P(\tau)$ has finite order, then p has odd order.*

Proof. Assume $p \in P(\tau)$ has an even order, since $< p > \subseteq P(\tau)$ by Theorem 2.2, there exists an element of order 2 in $P(\tau)$ contradicting the uniqueness of the root of 1. ▌

Theorem 3.3. *If G is a group with an involution τ such that every $p \in P(\tau)$ has odd order, then G has a polar decomposition.*

Proof. Since every $p \in P(\tau)$ has odd order by Euler's Theorem $p^{2^k} = p$ for some $k \geq 1$. So, $p^{2^{k-1}}$ is the unique square root of odd order of p and by Theorem 2.2 $p^{2^{k-1}} \in P(\tau)$. Thus every element in $P(\tau)$ has a unique square root in $P(\tau)$. So G has a polar decomposition. ▌

Corollary 3.2. *A group G has a polar decomposition if and only if for some involution $\tau \in Aut(G)$ every torsion element $p \in P(\tau)$ has odd order, and every nontorsion element has a unique root in $P(\tau)$.*

Corollary 3.3. *A torsion group G has a polar decomposition if and only if for some involution $\tau \in Aut(G)$ every $p \in P(\tau)$ has odd order if and only if every product of two conjugates of τ in G has odd order.*

Proof. Corollary 3.2, and Remark 2.1. ▌

Theorem 3.4. *For a locally finite group G. The group G has a polar decomposition, if and only if for some involution $\tau \in Aut(G)$, $P(\tau) \subseteq O(G)$.*

Proof. Corollary 3.2, and Theorem 3.2. ▌

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